Research Paper

The OLG Model Environment

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1   Introduction

1.1   Background and More Information

The large baby-boom cohort, which has affected economic growth for six decades, has just started to enter retirement. How will this baby-boom retirement affect growth and asset values over the next 30 years? The research project[1], *Population Aging, Implications for Asset Values, and Impact for Pension Plans: An International Study*, seeks to quantify the impact that population structure has on asset values and to project this impact on selected pension plans in Canada, the United Kingdom, and the United States.

This research program is a multi-year investigation involving many subprojects. An international partnership has been established between two universities, the University of Waterloo and the University of Kent, and three actuarial associations, the Society of Actuaries, the Institute and Faculty of Actuaries and the Canadian Institute of Actuaries. The Society of Actuaries has funded a subproject with respect to a literature review and specifications for the first stage of the model to be developed. The literature review has been provided as a separate document.

The purpose of this document is to provide information regarding the specifications of the model and to seek input from actuaries and others interested in this work. Interested parties may contact Doug Andrews at dwa007@hotmail.com.

This model's contribution is to generate returns to risky capital in the presence of demographic change with the extended modules producing insight into more complex demographic impacts. For example, gender differences in retirement patterns, life expectancy, risk tolerance and appetite, and household composition are documented in the literature and can have serious implications on income disparities both pre- and post retirement. However, these gender differences are often ignored in economic models due to the increased complexity inherent in their inclusion.

1.2   Introduction to the Model

In the following, we present a theoretical overlapping generations (OLG) model that will be one of the elements used to examine the impact of demographic change on asset class returns. An OLG model is a type of equilibrium growth model in which the economy contains one or more generations (cohorts) living in any given period of time. Each cohort lives for multiple periods and, as such, their life-span overlaps with those of other cohorts. Because the OLG framework

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allows for decision making that is specific to each life stage (e.g., education, fertility, work, and retirement), it is a useful tool for analysis of resource allocation across different generations.

Cohorts make decisions at each life stage. Some decisions are explicitly modelled. Thus, the outcomes of these decisions will be determined within the model. These decisions are referred to as endogenous. Other decisions are not incorporated into the model. For these decisions, we impose fixed behavioral constraints upon the cohorts, though they may be probabilistic in nature. These decisions that are not determined within the model are referred to as exogenous.

In a perfect world, the OLG model would have numerous types of household within each generation and many of the decisions facing each household would be endogenous to the model. (For example, a household that has children will face different opportunities and have different decisions than a household with no children). Unfortunately, intra-cohort heterogeneity makes the framework more complex, and can result in a model with no feasible solution. As such, there is a trade-off between the granularity or complexity of the model and its tractability.

With this trade-off in mind, the proposed model is comprised of a base model, together with extensions, or add-on modules. OLG models similar to our base model are utilized frequently in economic analysis. The add-ons are innovations to the literature and, as such, extending the models to include them and then solving these more complex models will be an exploratory process.

The base model has the following characteristics:

- one type of household representing each generation (no intra-cohort heterogeneity),
- endogenous labor supply,
- exogenous mortality,
- exogenous retirement probabilities,
- five overlapping generations,
- aggregate uncertainty on productivity, and
- two asset classes (risk-free and risky).

While rates of retirement and mortality are imposed on the model, different sets of assumptions will be explored to assess the sensitivity of the model's outputs to these assumptions. The five generations can be thought of as childhood, young working age, middle working age, old working age and retirement age. Labor supply is determined within the middle three cohorts by having the agents decide about labor force participation based on the trade-off between wage income and leisure.

The output from the model will be the following:
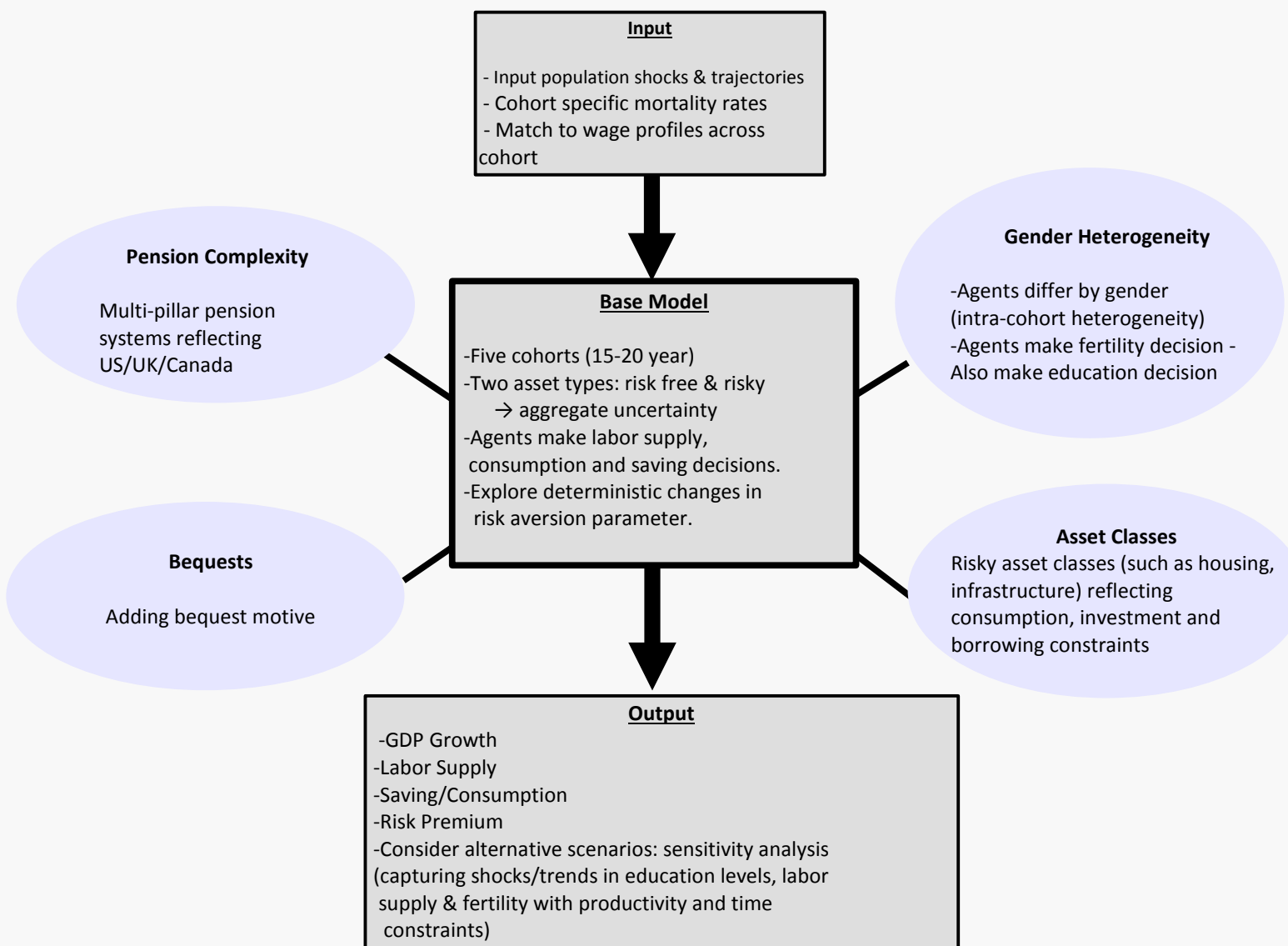
- GDP growth,
- aggregate savings,

- aggregate consumption,

- aggregate labor supply, and

- returns on assets.

The base model uses the aggregate data from the population, a weighted average of male-female characteristics. We can then perform sensitivity analysis of how changes in women's share of the population, labor market and fertility decisions/outcomes affect outputs over time, as the population ages. See Figure 1 (middle grey boxes).

Proposed extensions to the base model (blue ovals in Figure 1) include gender heterogeneity, increasing the complexity of proposed pension systems, adding bequests and analyzing other asset classes. Model extensions may have to be explored individually and/or within a simplified basic model. Each of the extensions, to our knowledge, would add significantly to our understanding of the repercussions of population aging.

A detailed theoretical description of the base OLG model is provided in next section. The appendix contains a glossary of terms.

Figure 1: Base Model (Grey) & Some Possible Extensions (Blue)

Input

- Input population shocks & trajectories
- Cohort specific mortality rates
- Match to wage profiles across cohort

Pension Complexity

Multi-pillar pension systems reflecting US/UK/Canada

Gender Heterogeneity

-Agents differ by gender (intra-cohort heterogeneity)
-Agents make fertility decision - Also make education decision

Base Model

-Five cohorts (15-20 year)
-Two asset types: risk free & risky
  → aggregate uncertainty
-Agents make labor supply, consumption and saving decisions.
-Explore deterministic changes in risk aversion parameter.

Bequests

Adding bequest motive

Asset Classes
Risky asset classes (such as housing, infrastructure) reflecting consumption, investment and borrowing constraints

Output
-GDP Growth
-Labor Supply
-Saving/Consumption
-Risk Premium
-Consider alternative scenarios: sensitivity analysis (capturing shocks/trends in education levels, labor supply & fertility with productivity and time constraints)

2   Basic Model

2.1   Demographics

The time period of the model is discrete. During each 20-year period, the household sector is made of 5 overlapping cohorts, of age between 0 and 99. We use $j \in \{0, 1, 2, 3, 4\}$ to denote cohort's age: 0: childhood; 1: young-working age; 2: middle-working age; 3: old-working age; 4: retirement age.

In this basic model, heterogeneity is intercohort only. That is, there is no heterogeneity within each cohort. Thus, we use a representive household j for each period t. The size of the household is given by $N_{j,t}$, which represents the size of cohort j in period t. Each period t, a new generation aged $j = 0$ (0-19 years old) is born into the economy, and the existing generations each shift forward by one life stage. The exogenous population growth rate of the new generation $j = 0$ in period t, is denoted by n_t, which we will hereafter refer to as the fertility rate in the model. Each household at age j has an exogenous marginal probability $\varphi_{j,t}$ of reaching age $j + 1$ in period $t + 1$. The oldest generation, $j = 4$, dies out deterministically in the subsequent period, i.e. $\varphi_{4,t} = 0$. Then, the population in period t is expressed as below:

$$N_{j,t} = \begin{cases} (1+n_t)N_{0,t-1}, & \text{if } j = 0, \\ \phi_{j-1,t-1}N_{j-1,t-1}, & \text{if } j \in \{1,2,3,4\}, \\ 0, & \text{if } j > 4, \end{cases}$$

(1)

And the population share of each cohort j, in period t is given by:

$$\mu_{j,t} = \frac{N_{j,t}}{\sum_{i=0}^{4} N_{i,t}}.$$

(2)

2.2   Households

Children, $j = 0$, are not active decision-makers. At each working age, each representative household (for $j = 1, 2, 3$) has a fixed constant H units of time to spend on labor and leisure. In addition, at the young and middle working ages $j = 1, 2$, the household mandatorily spends $FC_{j,t}$ units of time per period on fertility (which can be thought of as time required for child rearing). Moreover, at young working age $j = 1$, the representative household is required to take $FE_{j,t}$ units of time on education. So, we have $FC_{j,t} = 0$ if $j \notin \{1, 2\}$ and, $FE_{j,t} = 0$ if $j \neq 1$. Both $FC_{j,t}$ and $FE_{j,t}$ are exogenously given.

In each of the working ages, $j \in \{1, 2, 3\}$, the representative household supplies labor to the representative firm and earns a wage income according to their labor efficiency $\varepsilon_{j,t}$, which is exogenously given. Starting from $j = 3$ (age 60-79), the household receives some pension

income. Retirees (j = 4) supply zero labor and enjoy all available time as leisure with pension income.

Two financial assets act as a store of value over each period. Let $a_{j,t}$ and $a_{j,t}^r$ represent an age j household's holdings of the first (risk-free) and the second (risky) assets at the beginning of period t, respectively. Corresponding net returns on these two assets in period t are denoted by r_t and r_t^r. Young workers j = 1 enter the economy with zero asset holding, i.e. $a_{1,t} = a_{1,t}^r = 0$. Moreover, retirees leave the economy without any asset holding, i.e. $a_{5,t} = a_{5,t}^r = 0$.

Holding risk-free assets can be thought of as safe saving and may be negative, which reflects the fact that households may borrow. The net saving equals the government issued bonds, which is used to finance government spending and has to be positive.

$$B_{t+1} = \sum_{j=1}^{4} a_{j+1,t+1} N_{j,t}.$$

(3)
                                                                                          .

The second financial asset is risky in terms of aggregate uncertainty. In this basic model, we assume that the only aggregate uncertainty is from shocks on productivity, which will be discussed in detail later in the production subsection. Without loss of generality, a household's holdings of the risky asset could be thought of as the amount of capital they own. Note $a_{j,t}^r \geq 0$ and the summation equals the aggregate stock of capital in period t, K_t.

$$K_{t+1} = \sum_{j=1}^{4} a_{j+1,t+1}^r N_{j,t}.$$

(4)
                                                                                          .

We assume there is no bequest/inheritance motive in the basic model. If a household dies accidentally, its net wealth is collected by the government rather than being inherited. The government collects all residual assets from the fraction of the population that died and uses this as part of its general revenue.

The timing of the model is as follows. At the beginning of each period t, the household j's asset holdings are $a_{j,t}$ and $a_{j,t}^r$, which are brought from period t − 1. During the period, the household supplies labor to the firm and earns an income commensurate with their efficiency, hours and the market wage. At the end of period t, the household's total available resources include gross return on $a_{j,t}$ and $a_{j,t}^r$, wage income, and pension income, net of children's consumption expenditure $c_{0,t}$, less taxes. Then the household decides how to allocate these resources on consumption, $c_{j,t}$, and asset holdings to the next period, $\{a_{j+1,t+1}, a_{j+1,t+1}^r\}$. The aggregation of $\{a_{j+1,t+1}, a_{j+1,t+1}^r\}_{j=1}^{4}$ are used to finance newly issued government bonds

B_{t+1} and capital K_{t+1}. Deaths occur at the end of the period and the residual assets from the fraction of the population that died are collected by the government, denoted by ξ_t.

In each period, households maximize expected remaining lifetime utility subject to their respective time and budget constraints. Moreover, households are subject to several taxes. τ_t^c, τ_t^h and τ_t^p are exogenously given proportional taxes on consumption, labor income and pension income in period t, respectively. τ_t^s is the exogenous contribution rate for public pension. The gross (before tax) wage income of a household is the product of wage rate w_t and the amount of efficient labor $\varepsilon_{j,t} h_{j,t}$. At old working age $j = 3$, in addition to receiving the pension p_t^p, the household determines how much labor to supply out of the total $\iota_t^p H$ units of time. Thus, ι_t^p is the maximum fraction of the period that an old working household may work in period t.

2.3 Production

At each period, a representative firm uses labor H_t, in efficient units, and physical capital K_t to produce total final goods Y_t. We assume a Cobb-Douglas production function and no adjustment cost on capital:

$$Y_t = A_t K_t^\alpha H_t^{1-\alpha},$$

where $\alpha \in (0, 1)$ is the capital share and A_t is the total factor productivity.

The aggregate amount of efficient labor in period t, H_t, is given by:

$$H_t = \sum_{j=1}^{3} \varepsilon_{j,t} h_{j,t} N_{j,t}.$$

(5) .

The profit-maximizing behavior of the firm gives rise to first order decisions that determine the real net-of-depreciation rate of return to capital and the real wage rate per unit of efficient labor, respectively:

$$\textbf{(6) } r_t^r = \alpha A_t K_t^{\alpha-1} H_t^{1-\alpha} - \delta,$$

$$\textbf{(7) } w_t = (1-\alpha) A_t K_t^\alpha H_t^{-\alpha}.$$

where $\delta \in (0, 1)$ is the depreciation rate.

2.4 Social Security System

The social security system provides public pension p_t^p to the old working age household (at age $j = 3$) in period t. Note that a retiree $j = 4$ in period t receives public pension p_{t-1}^p. In each period t, the public pension p_t^p depends on the historical wage income of cohort $j = 3$ during their first two working periods: young and middle working ages.

The cost of the public pension benefits is covered fully by the social security system. The system adjusts the contribution rates for public pension, τ_t^s, so that budget balance is separately maintained for the public pension in every period.

2.5   Government

The government issues one-period risk-free bonds, receives residual assets from the fraction of the population that died, and collects taxes on consumption and labor to finance interest repayment on previously issued bonds and its spending G_t, which is exogenously given. Government bond issuance is adjusted so that the following consolidated budget constraint holds in every period:

$$(8)\ B_{t+1} + \xi_t + \tau_t^c C_t + \tau_t^h w_t H_t = G_t + (1 + r_t)B_t,$$

where B_t is government bonds at the beginning of period t and equals net household savings. C_t is period t aggregate consumption.

2.6   Recursive Competitive Equilibrium

At the beginning of each period, the state of the economy can be characterized by the state

$$\{\{a_{j,t}, a_{j',t}^r, \varphi_{j,t}, N_{j,t}, \mu_{j,t}, FC_{j,t}, FE_{j,t}\}_{j=0}^4, K_t, n_t, \xi_t, p_t^p, \ p_{t-1}^p, A_t, \varepsilon_{j,t}, \iota_t^p, \tau_t^c, \tau_t^h, \tau_t^p, G_t, B_t\}$$

Define the **Recursive Competitive Equilibrium** as sequences of prices $\{w_t, r_t, r_t^r\}_{t=0}^\infty$, allocations $\{C_t, H_t, K_{t+1}, B_{t+1}\}_{t=0}^\infty$, household decision rules $\{\{c_{j,t}\}_{j=1}^4, \{h_{j,t}, a_{j+1,t+1}, a_j^r{}_{+1,t+1}\}_{j=1}^3\}_{t=0}^\infty$, demographic structure $\{\{N_{j,t}\}_{j=0}^4\}_{t=0}^\infty$, pensions $\{p_t^p, \tau_t^s\}_{t=0}^\infty$, and residual assets from the fraction of the population that died $\{\xi_t\}_{t=0}^\infty$, such that, in each period, the following conditions are satisfied:

1. each cohort solves the utility maximizing problem;

2. the firm maximizes its profits, given prices;

3. the budget constraints of the social security system and the government hold;

4. the market clearing conditions hold for labor, capital, and government bonds;

5. the goods market clears.

Appendix

Glossary

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Table 1   *Parameters that may be calibrated*

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Parameter	Description
H	Total available time to spend for households
α	Capital share of production
δ	Depreciation rate of capital

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Table 2   *Exogenous variables drawn from data and projections*

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Exogenous Variable	Description
$c_{0,t}$	Consumption spending of children ($j = 0$)
n_t	Population growth rate, also the fertility rate
A_t	Total factor productivity (TFP)
$FC_{j,t}$	Units of time per period on fertility
$FE_{j,t}$	Units of time per period on education
G_t	Government spending
$\varepsilon_{j,t}$	Age- and time-specific labor efficiency
ι_t^p	Maximum fraction of one period that an old working household may work
τ_t^c	Proportional tax on consumption
τ_t^h	Proportional tax on labor income
τ_t^p	Proportional tax on pension income
$\varphi_{j,t}$	Suvival probability from age j to age $j + 1$

Table 3   *Endogenous variables generated by the model*

Endogenous Variables	Description
$a_{j,t}$	Risk-free asset holding at the beginning of period t
$a^r_{j,t}$	Risky asset holding at the beginning of period t
$c_{j,t}$	Consumption of cohort j in period t
$h_{j,t}$	Labor supply of cohort j in period t
p^p_t	Public pension to an old working age household in period t
r_t	Net return on risk-free asset
r^r_t	Net-of-depreciation return on capital
w_t	Wage rate per unit of efficient labor
B_{t+1}	Government bonds issued in period t
C_t	Aggregate consumption
H_t	Aggregate amount of efficient labor
K_t	Aggregate capital stock at the beginning of period t
$N_{j,t}$	Population size of cohort j
Y_t	Total final goods produced in period t
$\mu_{j,t}$	Population share of cohort j
ξ_t	Residual assets from the fraction of the population that died
τ^s_t	The contribution rates to balance social security