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Research Paper

LRMoE.jl: A Package for Insurance Loss Modelling Using a Mixture of Experts Regression Model

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Highlights of

LRMoE.jl: a software package for insurance loss modelling using a mixture of experts regression model

and

LRMoE: an R package for flexible actuarial loss modelling using a mixture of experts regression model

In recent years, we at the University of Toronto (Andrei L. Badescu, X. Sheldon Lin, and current and former PhD students) have been working on research projects on ratemaking and reserving for property/casualty (P&C) insurance. Our goal is to develop new and implementable technologies and ready-to-use software packages for actuaries in this area. This project is one of the results of these efforts. It was funded by an Academic Research Grant from the Canadian Institute of Actuaries (CIA), which we gratefully acknowledge.

In this project, we introduce new open-source statistical software tailor-made for actuarial applications which allows actuarial practitioners to model and analyze insurance loss frequencies and severities using a nonlinear multivariate regression model. The model is very flexible, and it can fit any type of positive dataset and capture the dependency structure implied by the data, and is statistically implementable.

Two papers with corresponding software packages are produced: a vignette for an R package and the other for a Julia package. R is known by many actuaries, while Julia is a highly efficient programming language that has been widely used by the machine learning and computer science community. The Julia package runs roughly four times faster than the R package. The packages can be downloaded at https://github.com/sparktseung/LRMoE.jl and https://github.com/sparktseung/LRMoE, respectively. They offer distinctive features which cannot be achieved using existing software packages. Key features include a wider coverage on frequency and severity distributions and their zero-inflated versions, parameter estimation under data censoring and truncation, and a collection of insurance ratemaking and reserving functions. The packages also provide several model-evaluation and model-visualization functions to help users easily analyze the performance of the fitted model and interpret the model in insurance contexts.

The underlying model and methodology developments of our software packages can be found in the following papers:

- Fung, T.C., Badescu, A., and Lin, X.S. (2019). “A class of mixture of experts models for general insurance: Application to correlated claim frequencies.” *ASTIN Bulletin*, **49**(3), 647–688.
- Fung, T.C., Badescu, A. and Lin, X.S. (2019). “A class of mixture of experts models for general insurance: Theoretical developments.” *Insurance: Mathematics and Economics*, **89**, 111–127.
- Fung, T.C., Badescu, A., and Lin, X.S. (2020). “A new class of severity regression models with an application to IBNR prediction.” *North American Actuarial Journal*, **25**(2), 1–26.

- Fung, T.C., Badescu, A., and Lin, X.S. (2021). “Fitting censored and truncated regression data using the mixture of experts models.” Available in SSRN: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3740061.

These papers are available upon request. If you have any comments, questions, or feedback, please contact either of us.

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LRMoE.jl: A package for insurance loss modelling using a mixture of experts regression model

Abstract

This paper introduces a new Julia package, **LRMoE**, statistical software tailor-made for actuarial applications which allows actuarial researchers and practitioners to model and analyze insurance loss frequencies and severities using the Logit-weighted Reduced Mixture of Experts (LRMoE) model. **LRMoE** offers several new distinctive features which are motivated by various actuarial applications and mostly cannot be achieved using existing packages for mixture models. Key features include a wider coverage on frequency and severity distributions and their zero inflation, the flexibility to vary classes of distributions across components, parameter estimation under data censoring and truncation, and a collection of insurance ratemaking and reserving functions. The package also provides several model evaluation and visualization functions to help users easily analyze the performance of the fitted model and interpret the model in insurance contexts.

Keywords: Multivariate regression analysis, censoring and truncation, Expectation-Conditional-Maximization Algorithm, insurance ratemaking and reserving, Julia.

1. Introduction

The Logit-weighted Reduced Mixture of Experts (LRMoE) model is a flexible regression model introduced by Fung et al. (2019b), which is regarded as the regression version of a finite mixture model with the mixing weights (called the *gating function*) which depend on the covariates. We may interpret the LRMoE as clustering policyholders into different subgroups with varying probabilities. Conditioned on the subgroup component to which each policyholder is assigned, the distributional properties of loss frequency or severity are governed by mixture component functions (called the *expert function*). Model flexibility, parsimony, and mathematical tractability are justified (see Fung et al. 2019b), demonstrating a sound theoretical foundation of LRMoE in a general insurance loss modelling perspective. Considering some specific choices of expert functions, Fung et al. (2019a) and Fung et al. (2020) construct Expectation-Conditional-Maximization (ECM) algorithms for efficient frequency and severity model calibrations and show the potential usefulness of LRMoE in terms of insurance ratemaking and reserving.

While the existing R package **flexmix** (Leisch 2004, and Grün and Leisch 2008) may perform parameter estimation for some special cases of LRMoE, it offers only limited choices of component functions (Poisson, Gaussian, Gamma, and Binomial) for model fitting. Miljkovic and Grün (2016) have used its extensibility feature to prototype new mixture models with alternative component functions (such as Lognormal, Weibull, and Burr), but users are still constrained to choosing a single parametric distribution for all the components.

This paper introduces a new package in Julia language, **LRMoE**, statistical software tailor-made for actuarial applications which allows actuarial researchers and practitioners to model and analyze insurance loss frequencies and severities using the LRMoE model. The package offers several new distinctive features which are motivated by various actuarial applications and generally cannot be achieved by existing packages, including:

- Fast fitting: With the increasing need to analyze large insurance datasets with hundreds of thousands of observations, it is crucial that a statistical package can fit models within a reasonable time frame. Compared with traditional languages such as R, our implementation in Julia significantly shortens the run time, allowing users to obtain and analyze results much faster (see Section 4.1 for comparison).
- Wider coverage on frequency and severity distributions: Apart from the severity distributions covered by Miljkovic and Grün (2016), the package also covers more frequency expert functions important for actuarial loss modelling, including negative binomial distribution and Gamma-count distribution.
- Zero-inflated distributions: Often actuaries are more interested in analyzing the aggregate loss for each policyholder instead of considering frequency and severity separately. In this situation, it is common practice to observe excessive zeros, which motivates the use of zero-inflated expert functions in the LRMoE, which is offered in this package. Note that efficient computation of zero-inflated LRMoE requires defining an additional latent variable (see Section 2.2 for details), further hindering the effectiveness of using the extensibility feature of **flexmix**.
- Package extensibility: In addition to providing a wide coverage of distributions, our package also allows users to define their customized expert functions with simple guidance from the package documentation. Hence, the package can be used not only within the actuarial community, but also in a wider range of research and practical problems.
- Varying classes of distributions across components: Insurance loss data may exhibit a mismatch between body and tail behaviours, which should be captured using different distributions. One approach is to choose two distributions and combine them using a peaks-over-threshold method (see, for example, Lee et al. 2012, and Scollnik and Sun 2012). Another is to consider a finite mixture model based on different component distributions (see, for example, Blostein and Miljkovic 2019). The **LRMoE** package is similar to the latter, and users can select different expert functions across different mixture components, which allows for more flexible and realistic modelling of data.
- Incomplete data: In many actuarial applications, including reinsurance, operational risk management, deductible ratemaking, and loss reserving, censored and truncated data are often observed and need to be dealt with. Censoring and truncation of LRMoE is introduced by Fung et al. (2021) with the expert functions restricted to univariate Gamma distribution. The new package removes such restriction by offering users the versatility to fit randomly censored and truncated multivariate data with many choices of expert functions.
- Model selection and visualization: In addition to the model-fitting function, the new package also provides several model-evaluation (AIC, BIC) and model-visualization (e.g., latent class probabilities, covariate influence) functions to help users easily analyze the performance of the fitted model and interpret the fitted model in the insurance context.

- Insurance ratemaking and reserve calculation: The package further contains a number of pricing and reserving functions (e.g., mean; variance; value-at-risk, or VaR; conditional tail expectation, or CTE), which enable actuaries to simultaneously perform ratemaking to multiple insurance contracts with different characteristics, based on abundant choices of premium principles.

The paper is organized as follows. Section 2 reviews the LRMoE model and parameter estimation using the ECM algorithm. In Section 3, we use a simulated dataset to demonstrate the basic fitting procedure in the **LRMoE** package. Section 4 contains more package utilities, such as parameter initialization, model visualization, and pricing function, which are illustrated using a French auto insurance dataset. The paper is concluded with some remarks in Section 5. For brevity, we only present code lines which are the most relevant to our new package. The source code, package documentation, and complete replication code for all examples in this paper are available at https://github.com/UofTActuarial/LRMoE.jl and https://uoftactuarial.github.io/LRMoE.jl/dev/. Further, we have developed a corresponding R package (accelerated by **Rcpp**) for fitting LRMoE with similar functionalities for users interested in running the package in R instead. We refer such readers to Tseung et al. (2021) for the vignette, and https://github.com/UofTActuarial/LRMoE for the code and documentations.

2. LRMoE Model and Parameter Estimation

In this section, we provide a brief overview of the LRMoE model proposed in Fung et al. (2019b), and discuss the ECM algorithm for parameter estimation. For brevity of presentation, we will assume, in sections 2.1 and 2.2, that all response variables (claim frequency or severity) are observed exactly. In Section 2.3, we will address data truncation and censoring for the LRMoE model.

2.1 Logit-weighted Reduced Mixture of Experts

Let $\boldsymbol{x}_i = (x_{i0}, x_{i1}, \ldots, x_{iP})^T$ denote the (*P* +1)-dimensional covariate vector for Policyholder *i* (*i* = 1, 2, . . . , *n*) with intercept $x_{i0} = 1$, and $\boldsymbol{y}_i = (y_{i1}, y_{i2}, \ldots, y_{iD})^T$ denote the *D*-dimensional vector of their response variables, which can be either claim frequency or severity. Let $\boldsymbol{X} = (x_1, x_2, \ldots, x_n)^T$ and $\boldsymbol{Y} = (y_1, y_2, \ldots, y_n)^T$ denote all covariates and responses for a group of *n* policyholders.

Based on the covariates, Policyholder *i* is classified into one of *g* latent risk classes by a logit-gating function

$$\pi_j(\boldsymbol{x}_i; \boldsymbol{\alpha}_j) = \frac{exp(\boldsymbol{\alpha}_j^T \boldsymbol{x}_i)}{\sum_{j'=1}^{g} exp(\boldsymbol{\alpha}_{j'}^T \boldsymbol{x}_i)}, \quad j = 1, 2, \ldots, g, \tag{1}$$

where $\boldsymbol{\alpha}_j = (\alpha_{j0}, \alpha_{j1}, \ldots, \alpha_{jP})^T$ is a vector of regression coefficients for latent class j.

Given the assignment of latent class $j \in \{1, 2, \ldots, g\}$, the response variables $\boldsymbol{y}_i = (y_{i1}, y_{i2}, \ldots, y_{iD})^T$ are conditionally independent. For $d = 1, \ldots, D$ and $i = 1, \ldots, n$,

the *d*-th dimension probability density function (or probability mass function) of y_{id} is given by

$$g_{jd}(y_{id};\delta_{jd},\psi_{jd}) = \delta_{jd}\mathbf{1}\{y_{id}=0\} + (1-\delta_{jd})f_{jd}(y_{id};\psi_{jd}) \qquad (2)$$

where $\delta_{jd} \in [0,1]$ is a parameter representing the probability of assigning an observation into a zero-inated component (i.e., a probability mass at zero), $\mathbf{1}\{y_{id}=0\}$ is the indicator function, and $f_{jd}(y_{id};\psi_{jd})$ is the density of some commonly used distribution with parameters ψ_{jd} for modelling insurance losses. Table 1 gives a list of parametric distributions supported by the **LRMoE** package. Note that a probability mass δ_{jd} at zero allows for more realistic modelling of insurance data which usually exhibit excess zeros. As a naming convention, we will refer to g_{jd} as a component distribution/expert function, and f_{jd} as its positive part, although claim frequency distributions (e.g., Poisson) also have some probability mass at zero.

Table 1: Distributions Supported by LRMoE

Root	Distribution	$f_{jd}(y)$	Parameters
Gamma	Gamma	$\frac{1}{\theta^m\Gamma(m)}y^{m-1}e^{-y/\theta}$	$m>0,\ \theta>0$
Lnorm	Lognormal	$\frac{1}{y\sigma\sqrt{2\pi}}\exp\left[-\frac{(\log y-\mu)^2}{2\sigma^2}\right]$	$\mu\in\mathbb{R},\ \sigma>0$
Invgauss	Inverse Gaussian	$\sqrt{\frac{\lambda}{2\pi y^3}}\exp\left[-\frac{\lambda(y-\mu)^2}{2\mu^2 y}\right]$	$\mu>0,\ \lambda>0$
Weibull	Weibull	$\frac{k}{\lambda}\left(\frac{y}{\lambda}\right)^{k-1}\exp\left[-\left(\frac{y}{\lambda}\right)^k\right]$	$k>0,\ \lambda>0$
Burr	Burr	$e^{-\lambda}\frac{\lambda^y}{y!}$	$k>0,\ c>0,\ \lambda>0$
Poisson	Poisson	$\binom{y+n-1}{n-1}p^n(1-p)^y$	$\lambda>0$
nbinom	Negative Binomial	$\int_0^{ms}\frac{1}{\Gamma(ys)}u^{ys-1}\exp\{-u\}du$	$n>0,\ 0<p<1$
gammacount	Gamma-count	$-\int_0^{ms}\frac{1}{\Gamma((y+1)s)}u^{(y+1)s-1}\exp\{-u\}$ du	$m>0,\ s>0$
ZI-root	All above	$g_{jd}=\delta_{jd}\mathbf{1}\{y=0\}+(1-\delta_{jd})f_{jd}$	$0<\delta_{jd}<1$

Under LRMoE, the conditional probability density function (or probability mass function) of 5 y_i given covariates x_i is

$$h(\boldsymbol{y}_i; \boldsymbol{x}_i, \boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\Psi}) = \sum_{j=1}^{g} \left[\pi_j(\boldsymbol{x}_i; \boldsymbol{\alpha}_j) \times \prod_{d=1}^{D} g_{jd}(y_{id}; \delta_{jd}, \psi_{jd}) \right] \tag{3}$$

where $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_g)^T$ is a $g \times (P+1)$-matrix of mixing weights with $\boldsymbol{\alpha}_g = (0, 0, \ldots, 0)^T$ representing the default class, $\boldsymbol{\delta} = (\delta_{jd})_{1 \le j \le g, 1 \le d \le D}$ is a $(g \times D)$-matrix of probability masses at zero by component and by dimension, and $\boldsymbol{\Psi} = \{\psi_{jd} : 1 \le j \le g, 1 \le d \le D\}$ is a list of parameters for the positive part f_{jd} by component and by dimension. Note that $\boldsymbol{\alpha}_g = (0, 0, \ldots, 0)^T$ ensures that the model is identifiable; that is, there is a one-to-one mapping between the regression distributions and the parameters (see Jiang and Tanner 1999, and Fung et al. 2019a). Note that the total number of parameters of the model is given by $(g-1) \times (P+1) + g \times D + \sum_{j=1}^{g} \sum_{d=1}^{D} \#\{\psi_{jd}\}$, where $\#\{\psi_{jd}\}$ is the length of ψ_{jd}

For a group of *n* policyholders, the likelihood function is given by

$$L(\boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\Psi}; \boldsymbol{X}, \boldsymbol{Y}) = \prod_{i=1}^{n} h(\boldsymbol{y}_i; \boldsymbol{x}_i, \boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\Psi}) = \prod_{i=1}^{n} \left\{ \sum_{j=1}^{g} \left[\pi_j(\boldsymbol{x}_i; \boldsymbol{\alpha}_j) \times \prod_{d=1}^{D} g_{jd}(y_{id}; \delta_{jd}, \psi_{jd}) \right] \right\}. \tag{4}$$

2.2 Parameter Estimation

For parameter estimation in finite mixture models, the Expectation-Maximization (EM) algorithm is commonly used (see, for example, Dempster et al. 1977, and McLachlan and Peel 2004). However, for the LRMoE model, the M-step requires maximization of a non-concave function over all elements of $\boldsymbol{\alpha}$. Fung et al. (2019a) thus use the ECM algorithm (Meng and Rubin 1993), which breaks the M-step into several substeps. The ECM algorithm implemented in the **LRMoE** package is described as follows.

Denote $\boldsymbol{\Phi} = (\boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\Psi})$ as the parameters to estimate. For $i = 1, 2, \ldots, n$, we introduce a latent random vector $\boldsymbol{Z}_i = (Z_{i1}, Z_{i2}, \ldots, Z_{ig})^T$, where $Z_{ij} = 1$ if $\boldsymbol{y}_i$ comes from the *j*-th component distribution and $Z_{ij} = 0$ otherwise. For $d = 1, 2, \ldots, D$, we further write $Z_{ij} = Z_{ijd0} + Z_{ijd1}$, where $Z_{ijd0} = 0$ and $Z_{ijd1} = 1$ if the *d*-th dimension of $\boldsymbol{y}_i$ comes from the positive part f_{jd} of the *j*-th component, and $Z_{ijd0} = 1$ and $Z_{ijd1} = 0$ if it comes from the zero inflation δ_{jd}. The complete-data loglikelihood function is given by

$$
\begin{aligned}
l^{\mathrm{com}}(\boldsymbol{\Phi};\boldsymbol{X},\boldsymbol{Y},\boldsymbol{Z}) &= \sum_{i=1}^{n}\sum_{j=1}^{g} Z_{ij}\left\{\log \pi_j(\boldsymbol{x}_i;\boldsymbol{\alpha}_j) + \sum_{d=1}^{D}\log g_{jd}(y_{id};\delta_{jd},\boldsymbol{\psi}_{jd})\right\} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{g} Z_{ij}\log \pi_j(\boldsymbol{x}_i;\boldsymbol{\alpha}_j) + \sum_{i=1}^{n}\sum_{j=1}^{g}\sum_{d=1}^{D}\{Z_{ijd0}\log\delta_{jd} + Z_{ijd1}\log(1-\delta_{jd})\} \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{g}\sum_{d=1}^{D} Z_{ijd1}\log f_{jd}(y_{id};\boldsymbol{\psi}_{jd}).
\end{aligned}
$$

For each $i = 1, 2, \ldots, n$, the random vector Zi follows a multinomial distribution with count 1 and probabilities $(\pi_1(\boldsymbol{x}_i;\boldsymbol{\alpha}_1), \pi_2(\boldsymbol{x}_i;\boldsymbol{\alpha}_2), \ldots, \pi_g(\boldsymbol{x}_i;\boldsymbol{\alpha}_g))$. Given $Z_{ij} = 1$, the conditional distribution of Z_{ijd0} is Bernoulli with probability δ_{jd}. Hence, at the *t*-th iteration, the posterior expectations of Z_{ij}, Z_{ijd0} and Z_{ijd1} are

$$
\begin{aligned}
z_{ij}^{(t)} &= \mathsf{E}\{Z_{ij}|\boldsymbol{\Phi}^{(t-1)},\boldsymbol{X},\boldsymbol{Y}\} = \mathsf{P}\{Z_{ij}=1|\boldsymbol{\Phi}^{(t-1)},\boldsymbol{X},\boldsymbol{Y}\} \\
&= \frac{\pi_j(\boldsymbol{x}_i;\boldsymbol{\alpha}_j^{(t-1)}) \times \prod_{d=1}^{D} g_{jd}(y_{id};\delta_{jd}^{(t-1)},\boldsymbol{\psi}_{jd}^{(t-1)})}{\sum_{j'=1}^{g}\pi_{j'}(\boldsymbol{x}_i;\boldsymbol{\alpha}_{j'}^{(t-1)}) \times \prod_{d=1}^{D} g_{j'd}(y_{id};\delta_{j'd}^{(t-1)},\boldsymbol{\psi}_{j'd}^{(t-1)})},
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
z_{ijd0}^{(t)} &= \mathsf{E}\{Z_{ijd0}|\boldsymbol{\Phi}^{(t-1)},\boldsymbol{X},\boldsymbol{Y}\} \\
&= \mathsf{P}\{Z_{ijd0}=1|\boldsymbol{\Phi}^{(t-1)},\boldsymbol{X},\boldsymbol{Y},Z_{ij}=1\} \times \mathsf{P}\{Z_{ij}=1|\boldsymbol{\Phi}^{(t-1)},\boldsymbol{X},\boldsymbol{Y}\} \\
&= \frac{\delta_{jd}^{(t-1)}\mathbf{1}\{y_{id}=0\}}{\delta_{jd}^{(t-1)}\mathbf{1}\{y_{id}=0\} + (1-\delta_{jd}^{(t-1)})F_{jd}(0;\boldsymbol{\psi}_{jd}^{(t-1)})} \times z_{ij}^{(t)},
\end{aligned}
\tag{7}
$$

and

$$
z_{ijd1}^{(t)} = z_{ij}^{(t)} - z_{ijd0}^{(t)} \tag{8}
$$

where F_{jd} is the cumulative distribution function of the positive part f_{jd}.

CM-Step

In the CM-step, we aim to maximize $Q(\boldsymbol{\Phi};\boldsymbol{\Phi}^{(t-1)},\boldsymbol{X},\boldsymbol{Y})$), which can be decomposed into three parts as

$$
Q(\boldsymbol{\Phi};\boldsymbol{\Phi}^{(t-1)},\boldsymbol{X},\boldsymbol{Y}) = Q_{\boldsymbol{\alpha}}^{(t)} + Q_{\boldsymbol{\delta}}^{(t)} + Q_{\boldsymbol{\Psi}}^{(t)} \tag{9}
$$

where

$$Q_{\alpha}^{(t)} = \sum_{i=1}^{n}\sum_{j=1}^{g} z_{ij}^{(t)} \log \pi_j(\boldsymbol{x}_i;\boldsymbol{\alpha}_j), \tag{10}$$

$$Q_{\delta}^{(t)} = \sum_{i=1}^{n}\sum_{j=1}^{g}\sum_{d=1}^{D}\left\{ z_{ijd0}^{(t)} \log \delta_{jd} + z_{ijd1}^{(t)} \log(1-\delta_{jd})\right\}, \tag{11}$$

and

$$Q_{\boldsymbol{\Psi}}^{(t)} = \sum_{i=1}^{n}\sum_{j=1}^{g}\sum_{d=1}^{D} z_{ijd1}^{(t)} \log f_{jd}(y_{id};\psi_{jd}). \tag{12}$$

To maximize $Q_{\alpha}^{(t)}$, we use the same conditional maximization as described in Fung et al. (2019a). We first maximize it with respect to $\boldsymbol{\alpha}_1$ with $\boldsymbol{\alpha}_j$ fixed at $\boldsymbol{\alpha}_j^{(t-1)}$ for $j = 2, 3, \ldots, g-1$. The next step is to maximize with respect to $\boldsymbol{\alpha}_2$ with updated $\boldsymbol{\alpha}^{(t)}$ and other $\boldsymbol{\alpha}_j$ fixed at $\boldsymbol{\alpha}_j^{(t-1)}$ for $j = 3, 4, \ldots, g-1$. The process continues until all $\boldsymbol{\alpha}$'s have been updated. For obtaining each $\boldsymbol{\alpha}_j^{(t)}$, the Iteratively Reweighted Least Square (IRLS) approach (Jordan and Jacobs 1994) is used until convergence.

For $Q_{\delta}^{(t)}$, each δ_{jd} can be updated using the following closed-form solution

$$\delta_{jd}^{(t)} = \frac{\sum_{i=1}^{n} z_{ijd0}^{(t)}}{\sum_{i=1}^{n}(z_{ijd0}^{(t)} + z_{ijd1}^{(t)})}. \tag{13}$$

The maximization of $Q^{(t)}$ can also be divided into smaller problems by component j and by dimension d. For each $j = 1, \ldots, g$ and $d = 1, \ldots, D$, the problem is reduced to maximize a weighted loglikelihood of an expert function where the weights of each observation are given by $z^{(t)}_{ijd1}$. For updating each $\boldsymbol{\psi}_{jd}^{(t)}$ therefore, closed-form solutions are only available for very special distributions (e.g., Poisson, Lognormal). Numerical optimization is used in most cases, especially when the observation $\boldsymbol{y}_i$'s are not observed exactly (see Section 2.3).

As discussed in McLachlan and Peel (2004), a mixture of severity distributions may have unbounded likelihood, which leads to spurious models with extremely large or small parameter values. In the **LRMoE** package, we adopt the same maximum a posteriori (MAP) approach as in Fung et al. (2020), which uses appropriate prior distributions to penalize the magnitude of fitted parameters (see Section 3). The rationale of including penalty functions is to avoid obtaining spurious models due to the unbounded nature of the loglikelihood function. The penalty itself should be small enough so that it results to negligible impacts on the fitted model. On the other hand, the penalty functions avoid parameters diverging to unreasonable values so that the fitted model would become more robust.

For more details regarding the rationales and executions, readers are recommended to refer to Section 4 of Fung et al. (2020).

2.3 LRMoE with Censoring and Truncation

Censoring and truncation are common in insurance datasets and need to be dealt with. For example, when a policy limit is applied, loss amounts above the limit will be recorded as the limit only, which creates right censoring of the complete loss data; when a policy deductible is applied, loss amounts below the deductible are not reported to the insurer, thus leading to left truncation.

Fung et al. (2021) have discussed the LRMoE model with censoring and truncation, where all component distributions are Gamma. For parameter estimation with data censoring and truncation, the ECM algorithm in Section 2.2 is slightly modified, with an additional E-step to remove the uncertainties arising from censoring and truncation. Since the main purpose of this paper is to demonstrate the application of the **LRMoE** package, we will omit the details and refer interested readers to the cited paper.

For all distributions included in it, the **LRMoE** package can perform parameter estimation in the presence of data truncation and censoring. Consequently, the user's input is slightly different from existing packages for mixture models. A detailed example on model fitting in our package is given in Section 3.

2.4 Ratemaking and Reserving in LRMoE

The model structure of LRMoE allows for easy computation of quantities relevant to actuarial ratemaking and reserving (see Fung et al. 2019b). At the policyholder level, the moments and common measures of dependence (e.g., Kendall's tau and Spearman's rho) of $\boldsymbol{y}_i$ can be computed in simple forms. The VaR and CTE can also be numerically solved without much difficulty. Various premium principles can be applied to price insurance contracts, including pure premium, standard deviation (SD) premium, limited expected value (LEV), and stop-loss (SL) premium. Risk measures can also be calculated for each individual policyholder (e.g., 99%-VaR). At the portfolio level, simulation can be conducted to obtain the distribution of the aggregate loss of all policyholders, which is useful for calculating the total loss reserve and premium calculation. The simulation process is facilitated by a data simulator included in our package (see Section 4.4).

3. Example: Simulated Dataset

In this section, we will demonstrate how to fit an LRMoE model using a simulated dataset which accompanies the package. The package and dataset can be installed and loaded as follows.

```
# Install and load package
> using Pkg, JLD2
> Pkg.add(url="https://github.com/UofTActuarial/LRMoE.jl")
> using LRMoE
# Load demo data
> @load "X_obs.jld2" X_obs
```

> *@load "Y_obs.jld2" Y_obs*

To address data truncation and censoring, the user's input of response ***Y*** is different from existing packages. For each dimension *d* of observation ***yi***, instead of a single numeric input, a quadruple $0 \le t_{id}^l \le y_{id}^l \le y_{id}^u \le t_{id}^u \le \infty$ is needed, where t_{id}^l and t_{id}^u are the lower and upper bounds of truncation, and y_{id}^l and y_{id}^u are the lower and upper bounds of censoring. The exact value of y_{id} lies between censoring bounds such that $y_{id}^l \le y_{id} \le y_{id}^u$. For a sample of size *n*, an *n x* (4*D*)-matrix is needed, where each *n* _ 4-block describes one dimension of ***Y***.

Sample rows of Y_obs in the demo dataset are shown as follows. These rows of data illustrate three possible scenarios of data truncation or censoring. For the first row, both dimensions of ***yi*** are observed exactly without truncation, so $t_{i1}^l = t_{i2}^l = 0$, $y_{i1}^l = y_{i1}^u = y_{i1}$, $y_{i2}^l = y_{i2}^u = y_{i2}$ and $t_{i1}^u = t_{i2}^u = \infty$.

For the second row, the second dimension of ***yi*** is truncated at 5 (e.g., by imposing a policy deductible) but not censored, so $t_{i2}^l = 5$. For the third row, the second dimension of ***yi*** is right-censored at 100 (e.g., by applying a policy limit) and the exact value of y_{i2} is unknown, so $y_{i2}^l = 100$ and $y_{i2}^u = \infty$.

> Y_obs[[1,6003,7847],:]

3 x 8 DataFrame

Row	Tl_1	Yl_1	Yu_1	Tu_1	Tl_2	Yl_2	Yu_2	Tu_2
1	0.0	6.0	6.0	Inf	0.0	89.0332	89.0332	Inf
2	0.0	8.0	8.0	Inf	5.0	37.4133	37.4133	Inf
3	0.0	7.0	7.0	Inf	0.0	100.0	Inf	Inf

Table 2: Description of Demo Dataset

Covariate[1]	**Name**	**Description**
x_{i0}	intercept	Constant 1
x_{i1}	sex	1 for male, 0 for female
x_{i2}	agedriver	Driver's Age: 20{80
x_{i3}	agecar	Car's Age: 0{10
x_{i4}	region	1 for urban, 0 for rural
Response	**Name**	**Description**
y_{i1}	Y[,1]	Claim count from business line 1
y_{i2}	Y[,2]	Claim severity from business line 2
Row Index	**Y[,1]**	**Y[,2]**
1–6000	No censoring or truncation	No censoring or truncation
6001–8000	No censoring or truncation	Left-truncated at 5[2]
8001–10000	No censoring or truncation	Right-censored at 100

[1] All covariates are generated independently and uniformly at random.
[2] The complete dataset (X, Y) contains 10,000 rows. As a result of left-truncating Y[,2], 163 rows of data are discarded, and the observed dataset (X_obs, Y_obs) has 9,837 rows only.

Table 3: True Model of Demo Dataset

Logit Regression Coefficients:		
$\boldsymbol{\alpha} = \begin{bmatrix} -0.50 & 1.00 & -0.05 & 0.10 & 1.25 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$		
Component Distributions:		
	$j = 1$	$j = 2$
$d = 1$	Poisson $(\lambda = 6)$	ZI-GammaCount $(\delta = 0.20, m = 30, s = 0.50)$
$d = 2$	LogNormal $(\mu = 4.0, \sigma = 0.30)$	InverseGaussian $(\mu = 20.0, \lambda = 20.0)$

Table 4: Fitted Model 1 of Demo Dataset

Logit Regression Coefficients:		
$\hat{\boldsymbol{\alpha}} = \begin{bmatrix} -0.4318 & 1.0688 & -0.0502 & 0.0951 & 1.1967 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$		
Component Distributions:		
	$j = 1$	$j = 2$
$d = 1$	Poisson $(\lambda = 6.016)$	ZI-GammaCount $(\delta = 0.206, m = 29.956, s = 0.489)$
$d = 2$	LogNormal $(\mu = 4.001, \sigma = 0.297)$	InverseGaussian $(\mu = 20.304, \lambda = 21.756)$

Table 5: Fitted Model 2 of Demo Dataset

<table>
<tr><td colspan="3">Logit Regression Coefficients:</td></tr>
<tr><td colspan="3">$$\tilde{\alpha} = \begin{bmatrix} -0.4023 & 1.0643 & -0.0499 & 0.0955 & 1.1788 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$</td></tr>
<tr><td colspan="3">Component Distributions:</td></tr>
<tr><td></td><td>$j = 1$</td><td>$j = 2$</td></tr>
<tr><td>$d = 1$</td><td>ZI-Poisson
$(\delta = 0.003, \lambda = 6.103)$</td><td>ZI-Poisson
$(\delta = 0.206, \lambda = 30.585)$</td></tr>
<tr><td>$d = 2$</td><td>Burr
$(k = 1.309, c = 5.232, \lambda = 58.799)$</td><td>Gamma $(k = 1.624, \theta = 12.398)$</td></tr>
</table>

To fit an LRMoE model, the user only needs to minimally specify the following: initial guess of logit regression coefficients (alpha_init) and what component distributions to use for each dimension and each component, as well as an initial guess of their parameters (comp_init).

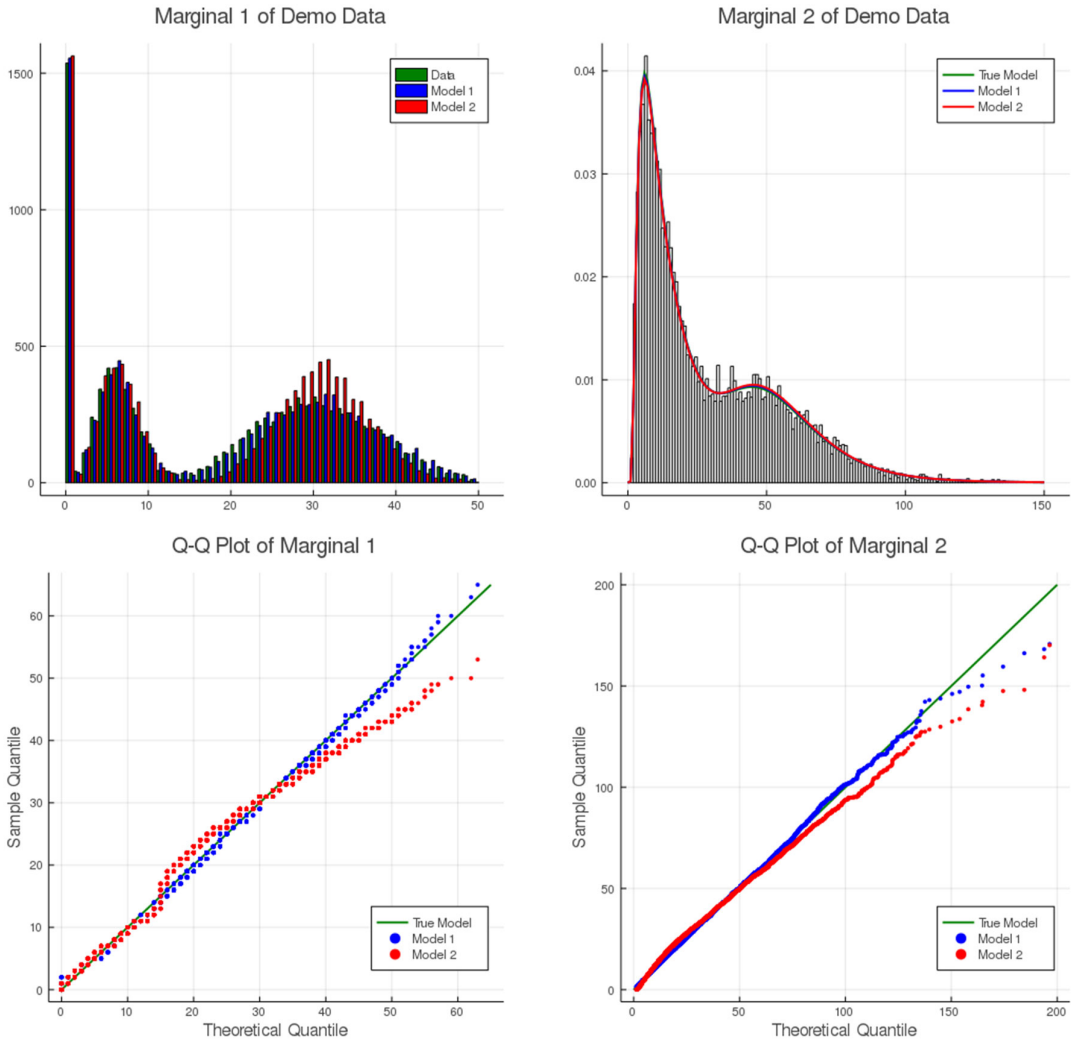
For illustration purposes, we first assume that the user's choice of component distributions coincides with the true model, and the initial guesses of parameters are also close to the true ones. The following sample code provides an example. With two components and five covariates, the initial guess alpha_init is a 2 × 5-matrix, where entries of zero indicate a non-informative guess. As for the component distributions, comp_init is a 2 × 2 matrix; here the number of rows (or columns) corresponds to the number of components (or dimension of response). Each entry of comp_init indicates a choice of expert function. In this case, y_{i1} is a mixture of Poisson with mean λ = 10.0 and zero-inflated Gamma-count distribution with zero inflation δ = 0.50, shape parameter m = 40, and dispersion parameter s = 0.80. Similarly, y_{i2} is a mixture of LogNormal(μ = 3.0, σ = 1.0) and InverseGaussian(μ = 15.0, λ = 15.0).

```
# Assume a non-informative guess
alpha_init = fill(0.0, 2, 5)

# Correctly specified component distributions
model_init = [PoissonExpert(10.0)          ZIGammaCountExpert(0.50, 40, 0.80);
              LogNormalExpert(3.0, 1.0) InverseGaussianExpert(15.0, 15.0)]
```

The fitting function of LRMoE can be called as follows, which will return a fitted model as well as loglikelihood and information criteria AIC and BIC. The result can be inspected by a summary() function, or by standard Julia methods.

```
# Call fitting function
result_1 = fit_LRMoE(Y_obs, X_obs, alpha_init, model_init)

# Result summary
summary(result_1) Model: LRMoE
Fitting converged after 7 iterations
Dimension of response: 2
Number of components: 2
```

Figure 1: Fitting results of DemoData



Loglik: -73153.32112999677
Loglik (no penalty): -73147.35233984645
AIC: 146320.7046796929
BIC: 146414.22545854456

 # Inspect fitted model
result_1.model_fit.alpha

2 x 5 Array{Float64,2}:

-0.431755	*1.06875*	*-0.0501667*	*0.0951105*	*1.19667*
0.0	*0.0*	*0.0*	*0.0*	*0.0*

2 result_1.model_fit.comp_dist
x 2 Array:

PoissonExpert(6.01676) *ZIGammaCountExpert(0.206196, 29.9564, 0.488854)*

LogNormalExpert(4.00105, 0.296734) *InverseGaussianExpert(20.3037, 21.7569)*

The fitted model is summarized in Table 4. The parameter estimates are quite close to the true ones. Considering simulated random noises and loss of information due to censoring and truncation, the fitting function is able to identify the true model when it is known.

In practice, when the true underlying model is not known, the user needs to perform some preliminary analysis on the dataset to determine model specification and parameter initialization. Table 5 contains the parameter estimates of another user-specified LRMoE model for the demo dataset, which has quite different component distributions compared with the true model.

The fitted loglikelihood values for Model 1 and Model 2 are -73147.35 and -74491.24, respectively. A graphical comparison of the two models is given in Figure 1. While both models have similar fitting performance for claim severity, Model 2 is noticeably worse in fitting small and extreme values of claim frequency.

4. Example: Real Dataset

In this section, we illustrate how to fit an LRMoE model to a real insurance dataset. As the basic fitting procedure has been discussed in the previous section, we will focus on other utilities of our package, including parameter initialization, model uncertainty, simulation, actuarial pricing functions, and model visualization.

4.1 Dataset and Computation Time

Throughout this section, we will use a French auto insurance claims dataset, freMTPLfreq and freMTPLsev, included in the R package **CASdatasets** (Dutang and Charpentier 2019). Exploratory analysis and data-cleaning procedures can be found on the accompanying website of our package. The cleaned dataset has 412,609 observations. The claim amount has high zero inflation, as less than 4% of policyholders have filed at least one claim. For positive claim amounts, the distribution is right-skewed, multi-modal, and heavy-tailed. The covariates used for modelling are described in Table 6.

Before proceeding, we remark on the computational time of our package. Compared with the demo dataset in Section 3, analyzing freMTPLfreq and freMTPLsev resembles a realistic actuarial modelling problem with many covariates and observations. In such a case, the Julia programming language is significantly more advantageous compared with traditional statistical languages such as R. Some standard benchmarks can be found on https://julialang.org/benchmarks/. Our experience confirms Julia's better performance: it takes around 20 hours in R (with optimization in **Rcpp**) to fit a model in this section, while Julia takes less than five hours to complete the same procedure.

For the corresponding R packages which we have developed, we refer readers to Tseung et al. (2021).

4.2 Parameter Initialization

Since the fitting procedure of LRMoE involves multivariate optimization, a good initialization of parameters will often lead to faster convergence, compared with a non-informative guess.

Table 6: Description of French Auto Insurance Data

Covariate	**Name**	**Description**
x_{i0}	Intercept	Constant 1. Default class for categorical variables
x_{i1}	CarAge	Vehicle age in years. Range: $0 \sim 100$
x_{i2}	DriverAge	Driver's age in years. Range: $18 \sim 99$
$x_{i3} \sim x_{i,13}$	Power	Power of the car as ordered categorical: $\mathrm{d} \sim \mathrm{o}$. Default is "d"
$x_{i,14} \sim x_{i,19}$	Brand	Brand of the car: 7 categories. Default is "Fiat"
$x_{i,20}$	Gas	Car gas: Diesel or Regular. Default is "Diesel"
$x_{i,21} \sim x_{i,29}$	Region	Policy region in France: 10 categories. Default is "Aquitaine"
Response	**Name**	**Description**
y_{i1}	ClaimAmount	Claim amount of the policyholder

Gui et al. (2018) have proposed an initialization procedure for a fitting mixture of Erlang distributions, which involves k-means clustering and the clusterized method of moments (CMM). This has been used in Fung et al. (2020) and offers reasonably good starting values of parameters.

Our package contains an initialization function which applies the CMM method to all component distributions. Some preliminary analysis is needed to determine the number of clusters (components) to use. Since the positive part of all distributions included in our package is unimodal, a heuristic starting point is to examine the empirical histogram of data and count the number of peaks (see Figure 3).

As an example, the procedure to initialize a three-component LRMoE is given as follows. The user needs to input response Y and covariate X, as used in the fitting function. In addition, the third argument 3 corresponds to the number of components, while the last argument *["continuous"]* indicates that the response Y is 1-dimensional and continuous. For the demo dataset used in Section 3, the input would be *["discrete" "continuous"]*.

```
# Initialize a 3-component model
init_3 = cmm_init(Y, X, 3, ["continuous"])
```

The cmm_init function will return a list of parameter initialization. For the logit regression coefficients, we assume a non-informative guess on covariate influence, resulting in zero coefficients on all covariates. The user is free to incorporate prior knowledge by modification afterwards. The relative size of each cluster is reflected by the intercept terms. In the following initialization, the size of three latent clusters is proportional to ($e^{-1.41667}$, $e^{-2.8687}$, $e^{0.0}$); that is, the proportion of these clusters within the entire dataset is given by (0.3842, 0.0330, 0.5827).

```
> init_3.alpha_init

3 x 30 Array{Float64,2}:
```

-1.41667 0.0 0.0 ... 0.0

-2.8687 0.0 0.0 ... 0.0

0.0 0.0 0.0 ... 0.0

As for the parameters of component distributions, for each dimension of Y we will apply the CMM method to obtain initialization for all types of expert functions. A summary of all initialization is given in Table 7. The initialization function cmm_init also returns summary statistics (mean, coefficient of variation, skewness, and kurtosis), which helps with choosing what combination of expert functions to use. Initialization with extremely large or small parameter values could result in a spurious model or bad fit, and should thus be avoided.

The cmm_init function also returns two suggested models based on the highest loglikelihood (ll_best) and the best Kolmogorov–Smirnov test (ks_best). For example, the model initialization with the highest loglikelihood is given by the following.

```
> init_3.ll_best

1 x 3 Array{ZILogNormalExpert{Float64},2}:
 ZILogNormalExpert{Float64}(0.960674, 6.89627, 1.05169)
 ZILogNormalExpert{Float64}(0.961532, 6.819891, 1.097471)
 ZILogNormalExpert{Float64}(0.958489, 6.80334, 1.09848)
```

4.3 Fitting Results and Model Selection

For illustration purposes, we fit only a selected number of LRMoEs to the entire French auto insurance dataset. The fitted loglikelihood, AIC, and BIC are summarized in Table 8. In most cases, adding more components will increase the fitted loglikelihood (e.g., consider models iwl, will, and wllil). The models selected by AIC and BIC have six and five components, respectively. In this particular setting, BIC heavily penalizes models with more components, since the sample size is large, and adding one component roughly increases the number of parameters by 30 (the number of logit regression coefficients).

Apart from AIC and BIC, cross-validation (CV) is an alternative model-selection criterion which avoids overfitting with too many latent components. For example, Gui et al. (2018) consider a 10-fold CV for a fitting mixture of Erlangs, where the averaged loglikelihood on the test sets is used as a score function to select the optimal number of components. CV can be implemented with the help of parallel computing in Julia (see also Section 4.5).

4.4 Pricing and Reserving Functions

Our package contains a collection of functions related to actuarial pricing, reserving, and risk management, including calculation of mean, variance, VaR, CTE, LEV $E[(Y \wedge u)]$, and SL premium $E[(Y - d)_+]$ of the response variable. These functions start with root predict_, followed by appropriate quantities of interest (mean, var, VaR, CTE, limit, excess) and corresponding function arguments.

As an illustration, we consider three policyholders in Table 9, where A has no claim history, B has a medium-sized claim, and C has a large claim. Below is the sample code to calculate different quantities of interest.

 # Mean of claim amount of Policyholders A, B and C.

> predict_mean(X[[1, 33, 96],:], alpha_fit, comp_fit)

[52.859, 52.577, 56.368]

Table 7: Sample Initialization of Parameters

Component		1	2	3
Proportion		0.3842	0.0330	0.5827
Zero Inflation		0.9615	0.9584	0.9607
Positive Data:				
- Mean		1686.01	1622.73	1680.57
- Coefficient of Variation		2.51	2.50	2.07
- Skewness		12.15	13.57	11.61
- Kurtosis		185.67	230.49	183.04
- Parameter Initialization				
Gamma	k	0.16	0.16	0.23
	λ	10616.28	10146.28	7231.11
LogNormal	μ	6.82	6.80	6.90
	σ	1.10	1.10	1.05
Inverse Gaussian	μ	1868.01	1622.73	1680.57
	λ	267.76	259.53	390.58
Weibull[1]	k	1.00	1.00	1.00
	θ	1686.01	1622.73	1680.57
Burr[1]	k	351.85	126.52	40.00
	c	0.88	0.90	0.97
	λ	1218490	317504	69819

[1] Since the moments cannot be written in closed form, these parameters are solved by maximizing loglikelihood based on the observation in clusters.

Table 8: Fitting Results of French Auto Insurance Data

***g* = 3**	**Loglikelihood**	**AIC**	**BIC**	***g* = 4**	**Loglikelihood**	**AIC**	**BIC**
lll (69)	-190010	380157	380991	lllb (103)	-189676	379558	380684
lib (70)	-190192	380524	381289	lill (102)	-189884	379973	381087
iwl (69)	-190110	380358	381112	llll (102)	-190008	380220	381335
llb (70)	-190202	380545	381310	will (102)	-190057	380319	381445
wll (69)	-190225	380588	381342	bill (103)	-190109	380422	381537
***g* = 5**	**Loglikelihood**	**AIC**	**BIC**	***g* = 6**	**Loglikelihood**	**AIC**	**BIC**
lllll (135)	-189323	378916	**380392**	llllll (168)	**-189268**	**378872**	380708
lllil (135)	-189662	379594	381070	llblll (169)	-189441	379220	381067
bllil (136)	-189696	379663	381150	liwlll (168)	-189939	380214	382050
bwliw (136)	-189767	379805	381292	llwlll (168)	-190235	380805	382642
wllil (135)	-189903	380076	381551	liwlbl (169)	-190276	380890	382738

Notes: Component distributions are represented by first letter; for example, l for lognormal, b for Burr. All components are zero-inated. The number in brackets is the total number of parameters. Stopping criterion is < 0:05 improvement in loglikelihood, or 500 iterations. For each *g*, loglikelihood values are in decreasing order. The overall optimal values are bolded.

 # 99% VaR of claim amount of Policyholders A, B and C.

> *predict_VaR(X_obs[[1, 33, 96],:], alpha_fit, comp_fit, 0.99)*

Table 9: Selected Policyholders in French Auto Insurance Dataset

	Claim	ID	Car Age	Driver's Age	Car Power	Brand	Gas	Region
A	0	1	0	46	*g*	JK	Diesel	A
B	302	33	1	64	*g*	JK	Regular	IF
C	9924	96	0	51	*j*	JK	Regular	IF

Table 10: Pricing Calculation for Selected Policyholders in French Auto Insurance Dataset

Policyholder Premium Principle		A		B		C	
		Prior	Posterior	Prior	Posterior	Prior	Posterior
Individual-level	$E[Y]$	52.86	51.99	52.58	215.42	56.37	200.28
	$E[(Y-1000)_+]$	29.83	29.29	28.51	149.53	30.44	145.76
	$E[Y \wedge 100000]$	39.22	38.46	39.67	162.63	43.15	140.00
Portfolio-level	VaR(90)	55.89	54.99	55.59	227.83	59.60	211.81
	VaR(95)	57.20	56.28	56.90	233.16	61.00	216.77
	CTE(70)	55.77	54.87	55.48	227.34	59.47	211.36
	CTE(80)	56.59	55.68	56.29	230.69	60.35	214.46
	CTE(90)	58.04	57.10	57.73	236.59	61.89	219.95

Notes: (for portfolio-level premium principles, the allocation is based on the relative size of pure premium, where the weight for Policyholder *i* is $w_i = E[Y_i]/\sum_j E[Y_j]$. For each policyholder and premium principle, the calculation is done based on both prior probability (left column) and posterior probability (right column))

[1198.970, 1195.295, 1204.096]

 # Mean excess of claim amount (d=1000) of Policyholders A, B and C.

> *predict_excess(X_obs[[1, 33, 96],:], alpha_fit, comp_fit, 1000)*

[29.825, 28.512, 30.435]

Actuarial pricing calculation can be done based on either individual loss distributions or the aggregated loss distribution of the entire portfolio. Both can be achieved using built-in functions in our package.

On the individual level, the functions above can be called directly to calculate the pure premium $E[Y]$, as well as the premium value in the presence of a policy deductible ($E[(Y - d)_+]$) or policy limit ($E[(Y \wedge u)]$). The first part of Table 10 summarizes the premium calculation for policyholders A, B, and C, based on individual-level premium principles.

On the portfolio level, the sim_dataset function (see also Section 4.6) will simulate one response value (i.e., claim amount) for each individual policyholder, which can be summed up to the aggregated portfolio loss under one possible scenario. Repeated simulation of the entire portfolio will produce the empirical distribution for the aggregated loss. The VaR and CTE of the aggregated loss can be obtained from the simulated sample, which are useful for setting the insurer's total reserve. In addition, the VaR and CTE can be allocated back to policyholders as a loaded premium, according to some weighting scheme which reflects policyholders' relative riskiness (e.g., relative magnitude of their pure premium). The second part of Table 10 summarizes the premium calculation for policyholders A, B, and C, based on portfolio-level premium principles.

4.5 Parameter Uncertainty

In addition to obtaining point estimates of model parameters, it is crucial to also calculate their confidence intervals in order to identify significant parameters. Given that the loglikelihood in Equation (4) becomes much more complicated when data are truncated

and/or censored, analytically deriving the variance and confidence intervals of parameter estimates would not be feasible, and could be subject to numerical issues in implementation.

Instead, we can use bootstrapping methods in Grün and Leisch (2004). A straightforward nonparametric bootstrapping algorithm is described as follows:

1. Fit an LRMoE using the original dataset, and obtain an estimate $\hat{\Phi}$.
2. For a fixed number of total iterations, say 200, sample with replacement from the original dataset, on which to fit an LRMoE again with the same component distributions.
3. The estimates $\hat{\Phi}$ 1, $\hat{\Phi}$ 2, . . . , $\hat{\Phi}$ 200 from the bootstrapped samples will provide an estimate of the variance of parameters.

The algorithm above can be easily implemented in parallel in Julia as follows. We have included the estimated confidence intervals in the Appendix, calculated based on 200 bootstrapped samples.

```
> using Distributed
> @Distributed for i in 1:200
    result = fit_LRMoE(Y, X, alpha_init, model_init)
    @save "result_"*"$(i)"*".JLD2" result # save the result end
```

4.6 Model Simulation

In the LRMoE setting, the loss distributions of policyholders are mixtures of the same expert functions, but with potentially different mixing weights. Consequently, the distribution of the aggregate loss, as a sum of individual losses, usually do not admit a simple form.

Our package contains a dataset simulator, which helps with analyzing the distribution of aggregated loss. Given a portfolio of policyholders X and a model specification alpha and comp_dist, the simulator will return one set of random realization of losses for each policyholder.

```
> sim_dataset(alpha, X, comp_dist)
```

With multiple calls to sim_dataset, an approximation of the aggregated loss can be obtained. This has been applied in Section 4.4 to calculate premium based on portfolio-level premium principles.

4.7 Model Visualization

After fitting and choosing an appropriate LRMoE model, the user can visualize it with in-package plotting functions, or create more customized plots using generic simulation functions combined with plotting utilities or other dedicated packages such as **Plots.jl** and **StatsPlots.jl**.

In this subsection, we will use the six-component llllll model for demonstration.

Latent Class Probabilities

The logit regression in LRMoE assigns each policyholder into latent risk classes based on

covariates. Given a fitted mode and a vector of covariates, the probability of latent classes can be computed and visualized using the built-in predict_class_prior and plot_class_prob functions.

Consider the policyholders in Table 9. The predict_class_prior function will return both latent class probabilities (prob) as well as the most likely class (max_prob_idx). The following sample code calculates their latent class probabilities, where each row represents a policyholder and each column represents a latent class. Based solely on covariates information, there is not a large difference between these policyholders, and all are most likely coming from the first latent class.

```
# Predict latent class probabilities, based on covariates and a model
> predict_class_prior(X[[1, 33, 96],:], alpha_fit).prob
```

```
0.400565  0.120057  0.0192732  0.00377685  0.295405  0.160923
0.450407  0.166925  0.0226136  0.00350641  0.204941  0.151607
0.490915  0.139407  0.0325873  0.00430055  0.179139  0.153652
```

```
# Predict latent class probabilities, based on covariates and a model
> predict_class_prior(X[[1, 33, 96],:], alpha_fit).max_prob_idx
[1, 1, 1]
```

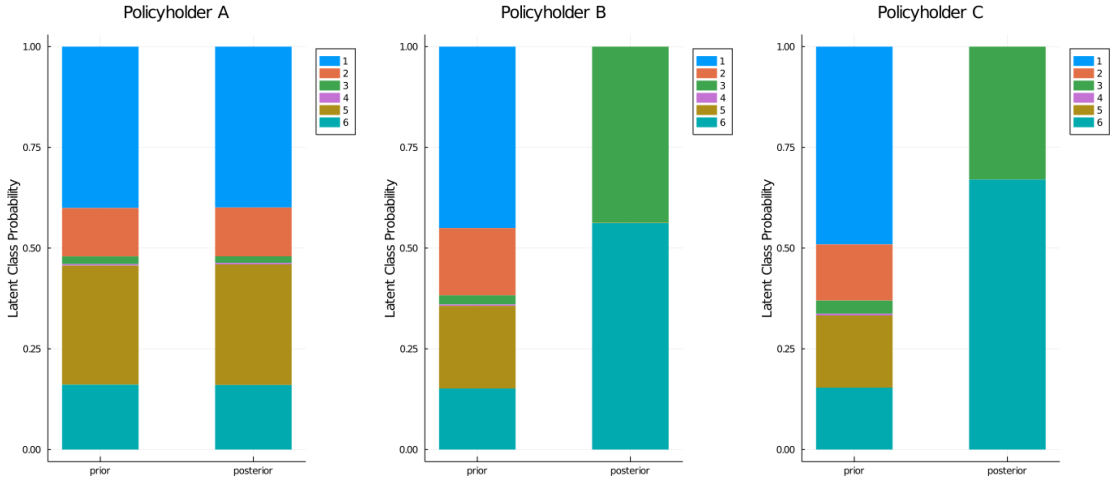
For policyholders with known claim history, it may be more informative to consider the posterior latent class probabilities by calling corresponding functions for posterior probabilities. The posterior probability of latent class j is given by $P\{Z_{ij} = 1 | \hat{\Phi}, \boldsymbol{X}, \boldsymbol{Y}\}$ for fitted parameters $\hat{\Phi}$ which can be computed analogous to Equation (6). Readers may also refer to Section 6.3.2 of Fung et al. (2019a) for more details.

Consider again the same policyholders in Table 9. The code to calculate posterior probabilities is similar as above. Given that policyholders B and C have non-zero claims, a lot of latent class probability is shifted to classes 3 and 6, which correspond to the middle and tail parts of the loss distribution (see also the largest spike in Figure 3). Meanwhile, Policyholder A has no claim, resulting in little change to the latent class probabilities since all components have very similar zero inflation.

The posterior probabilities are also helpful for adjusting premium rates. In Table 10, we contrast premium calculation based on both prior and posterior probabilities. It is clear that Policyholder A is rewarded by a decrease in premium for having no claim history, while B and C have significant increases in premium rates.

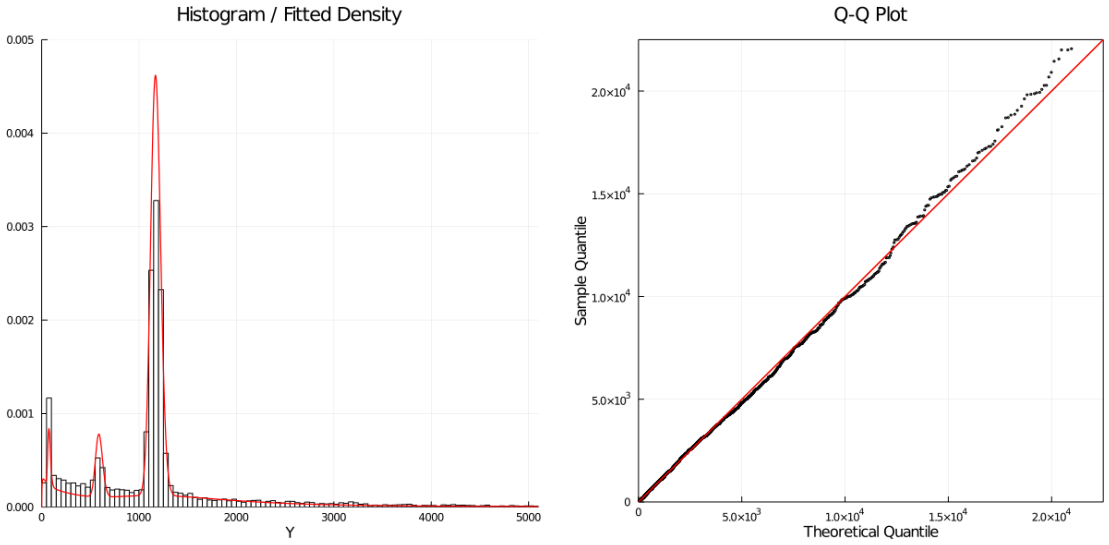
```
# Predict latent class probabilities, based on covariates and a model
> predict_class_posterior(Y[[1, 33, 96],:], X[[1, 33, 96],:],
                          alpha_fit, comp_fit).prob
```

Figure 2: Prior and posterior latent class probabilities for selected policyholders



Notes: (Component 6 (Lognormal(6.58, 1.95)) corresponds to the tail, while Component 3 (Lognormal(6.95, 1.05)) corresponds to the largest spike in the dataset)

Figure 3: Overall goodness-of-fit of positive claims in the French auto insurance dataset



0.399295	0.121097	0.0165664	0.00351454	0.299413	0.160114
9.7166e-207	1.36555e-25	0.437268	4.50508e-13	0.000754288	0.561978
0.0	0.0	0.329656	2.77853e-153	0.000165016	0.670179

Notes: (Left: all data points; right: non-extreme data points)

 # Predict latent class probabilities, based on covariates and a model

> predict_class_posterior(Y[[1, 33, 96],:], X[[1, 33, 96],:],alpha_fit, comp_fit).max_prob_idx

[1, 6, 6]

Overall Goodness-of-Fit

The overall goodness-of-fit can be examined by contrasting the empirical histogram against fitted density curve and the Q-Q plot of empirical and fitted quantiles. These can be produced with the sim_dataset function included in our package, combined with basic plotting functions in Julia. The corresponding plots are shown in Figure 3.

Covariate Influence

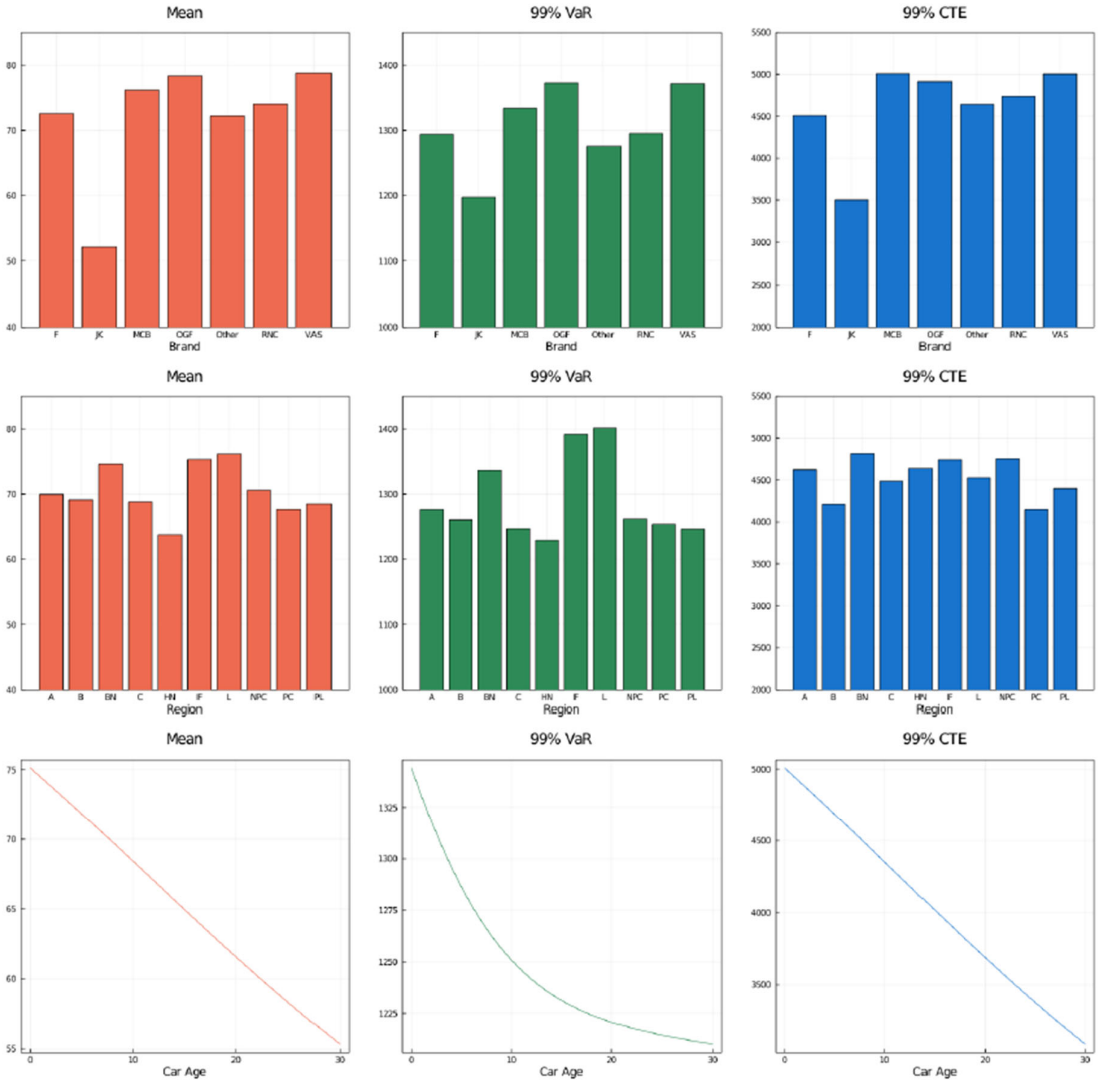
The partial dependence plot is commonly used in machine learning to investigate the influence of a particular covariate on the response, assuming independence among covariates (Friedman 2001). For example, the marginal effect of a covariate on the mean claim amount can be obtained using the following steps:

1. Fix the covariate at a particular value (say, car brand = "F") for all policyholders, while other covariates remain unchanged;
2. Use predict_mean_prior function to compute mean response values by policyholder (see Section 4.4);
3. Compute the grand mean of all values in step (2), which averages out the effect of other covariates and yields the mean response value when the car brand is of type "F";
4. Repeat steps (1)–(3) for a range of values of the same covariate of interest; for example, varying car brand from "F" to "VAS".

The procedure above can be generalized to continuous covariates and other quantities of interest, such as the quantile of the response variable. The covariate influence plot graphically illustrates how the characteristics of a policyholder are associated with a particular measure of the response variable. For example, plotting the mean (or VaR/CTE) of the response variable against car brand reflects how a policyholder's overall riskiness (or tail risk) is related to the car brand.

Figure 4 shows the influence of covariates Brand, Region, and Car Age, which may be interpreted as follows. For car brand, MCB (Mercedes, Chrysler, or BMW), OGF (Opel, General Motors, or Ford) and VAS (Volkswagen, Audi, Skoda, or Seat) are generally riskier. Also, compared with other regions, policies issued in regions IF (Île-de-France) and L (Limousin) may be considered riskier. In addition, older cars are generally associated with lower risks.

Figure 4: Covariate influence – Brand, Region, and Car Age



5. Summary and Outlook

This paper presented a new Julia package, **LRMoE**, for actuarial loss modelling. In this first version, the package can cover the most basic need to fit an LRMoE model, as well as helping the user visualize the fitted model and calculate the insurance premium.

There are several future developments in our plan, including:

- Model-selection tools: As of the current version, model selection is done by AIC/BIC/CV, all of which require the user to choose and run a selection of models. Some automated model-selection procedures may be potentially incorporated in the package (e.g., SCAD-type penalty in Fan and Li 2001, and Yin and Lin 2016).
- Feature-selection tools: Datasets usually contain a large number of covariates, but not all are important for predicting the response. While plots of covariate influence may offer some intuition of their relative importance, it may be more insightful to quantify their influence and provide a function to automatically choose the most influential covariates.

Furthermore, it is crucial to compare the fitting and predictive performances between our proposed LRMoE and classical regression models. Several existing papers have already shown that Erlang Count LRMoE (EC-LRMoE) performs much better than Negative Binomial GLM (see Section 6.1 (Table 7) of Fung et al. 2019a), and Transformed-Gamma LRMoE (TG-LRMoE) performs much better than various severity GLM (see Section 5.3.1 (Figure 2 and Table 2) of Fung et al. 2020). Nonetheless, it is still desirable to conduct extensive comparisons between LRMoE under different expert functions and a wider range of classical models (including, for example, GAM), and this will be addressed in a new paper that we are currently working on.

6. Appendix

The Appendix contains the parameter estimates of model llllll presented in Section 4, as well as their 95% confidence intervals. Significant parameters are marked in bold.

$\hat{\alpha}$	Class 1	Class 2	Class 3
Intercept	**0.733** (0.538, 0.916)	**-1.038** (-1.221, -0.675)	**-0.496** (-0.74, -0.243)
Car Age	**0.039** (0.031, 0.043)	**-0.015** (-0.039, -0.011)	-0.001 (-0.013, 0.003)
Driver Age	**0.009** (0.006, 0.01)	-0.001 (-0.006, 0.002)	-0.002 (-0.007, 0)
Power: e	0.047 (-0.012, 0.192)	-0.099 (-0.23, 0.129)	**0.201** (0.134, 0.368)
Power: f	-0.050 (-0.1, 0.071)	-0.007 (-0.123, 0.253)	**0.195** (0.098, 0.375)
Power: g	**-0.109** (-0.2, -0.021)	**-0.231** (-0.421, -0.061)	-0.031 (-0.166, 0.098)
Power: h	0.040 (-0.083, 0.176)	-0.228 (-0.437, 0.008)	0.091 (-0.077, 0.284)
Power: i	0.088 (-0.021, 0.27)	-0.240 (-0.488, 0.039)	**0.267** (0.102, 0.548)
Power: j	0.091 (-0.022, 0.259)	**-0.449** (-0.791, -0.17)	**0.297** (0.08, 0.472)
Power: k	0.053 (-0.101, 0.315)	-0.001 (-0.217, 0.344)	**0.305** (0.089, 0.654)
Power: l	-0.124 (-0.402, 0.119)	**-0.506** (-0.895, -0.077)	-0.016 (-0.467, 0.361)
Power: m	-0.071 (-0.503, 0.32)	**-0.546** (-0.863, -0.088)	-0.062 (-0.621, 0.436)
Power: n	**-0.577** (-1.083, -0.155)	0.078 (-0.657, 0.747)	-0.032 (-0.655, 0.501)
Power: o	-0.228 (-0.577, 0.355)	-0.466 (-0.973, 0.099)	0.192 (-0.603, 0.8)
Brand: JK	**-0.121** (-0.403, -0.07)	**1.023** (0.663, 1.16)	**-1.493** (-1.813, -1.269)
Brand: MCB	**-0.381** (-0.502, -0.173)	**-0.404** (-0.786, -0.103)	-0.231 (-0.443, 0.039)
Brand: OGF	-0.138 (-0.26, 0.024)	**-0.299** (-0.539, -0.04)	0.066 (-0.115, 0.248)
Brand: Other	-0.181 (-0.391, 0.03)	**-0.422** (-0.778, -0.066)	-0.259 (-0.514, 0.006)
Brand: RNC	**-0.189** (-0.306, -0.031)	-0.043 (-0.237, 0.201)	**-0.178** (-0.335, -0.003)
Brand: VAS	**-0.220** (-0.342, -0.036)	-0.219 (-0.45, 0.127)	-0.020 (-0.167, 0.176)
Gas: Regular	**-0.170** (-0.251, -0.148)	**-0.097** (-0.259, -0.018)	**-0.240** (-0.338, -0.174)
Region: BN	0.066 (-0.075, 0.284)	0.067 (-0.216, 0.396)	**0.248** (0.008, 0.531)
Region: B	**0.466** (0.39, 0.683)	**-0.317** (-0.583, -0.055)	**0.341** (0.194, 0.587)
Region: C	**0.111** (0.028, 0.262)	0.048 (-0.08, 0.257)	-0.040 (-0.159, 0.179)
Region: HN	**-0.464** (-0.808, -0.332)	0.106 (-0.372, 0.307)	**-0.745** (-1.233, -0.396)
Region: IF	**0.174** (0.126, 0.357)	**0.517** (0.426, 0.79)	**0.494** (0.361, 0.747)
Region: L	**0.514** (0.315, 0.855)	0.242 (-0.118, 0.78)	**0.763** (0.363, 1.185)
Region: NPC	-0.102 (-0.25, 0.018)	0.120 (-0.123, 0.306)	-0.176 (-0.421, 0.05)
Region: PL	**0.194** (0.078, 0.335)	0.086 (-0.082, 0.362)	0.036 (-0.188, 0.245)
Region: PC	**0.449** (0.343, 0.712)	0.144 (-0.078, 0.583)	**0.332** (0.18, 0.638)
δ	**0.970** (0.969, 0.970)	**0.981** (0.980, 0.984)	**0.836** (0.822, 0.840)
$\hat{\mu}$	**7.065** (7.064, 7.067)	**6.379** (6.370, 6.387)	**6.950** (6.892, 7.031)
$\hat{\sigma}$	**0.044** (0.042, 0.045)	**0.061** (0.053, 0.068)	**1.046** (0.958, 1.109)

$\hat{\alpha}$	Class 4	Class 5	Class 6
Intercept	**-1.362** (-1.845, -0.902)	-0.077 (-0.095, 0.327)	0
Car Age	-0.010 (-0.029, 0.004)	**-0.012** (-0.029, -0.011)	0
Driver Age	**-0.015** (-0.022, -0.011)	**-0.002** (-0.005, -0.001)	0
Power: e	**0.275** (0.035, 0.561)	-0.026 (-0.091, 0.091)	0
Power: f	**0.248** (0.031, 0.529)	**-0.082** (-0.217, -0.03)	0
Power: g	0.108 (-0.165, 0.343)	-0.029 (-0.117, 0.034)	0
Power: h	0.098 (-0.328, 0.463)	-0.014 (-0.21, 0.079)	0
Power: i	-0.009 (-0.413, 0.369)	**-0.313** (-0.619, -0.257)	0
Power: j	0.135 (-0.335, 0.584)	**-0.207** (-0.425, -0.109)	0
Power: k	0.483 (0, 1.003)	**-0.296** (-0.523, -0.151)	0
Power: l	0.518 (-0.146, 1.14)	-0.154 (-0.478, 0.091)	0
Power: m	0.134 (-0.786, 0.796)	-0.133 (-0.504, 0.156)	0
Power: n	0.560 (-0.528, 1.085)	-0.311 (-0.769, 0.043)	0
Power: o	0.559 (-0.453, 1.502)	**-0.528** (-0.966, -0.319)	0
Brand: JK	**-1.794** (-2.387, -1.432)	**0.797** (0.728, 1.025)	0
Brand: MCB	0.016 (-0.391, 0.357)	**0.254** (0.134, 0.453)	0
Brand: OGF	0.093 (-0.234, 0.465)	**-0.128** (-0.347, -0.01)	0
Brand: Other	0.100 (-0.401, 0.528)	0.087 (-0.085, 0.339)	0
Brand: RNC	-0.287 (-0.562, 0.034)	0.005 (-0.18, 0.091)	0
Brand: VAS	-0.304 (-0.636, 0.108)	-0.051 (-0.236, 0.066)	0
Gas: Regular	-0.035 (-0.207, 0.079)	**0.080** (0.019, 0.148)	0
Region: BN	0.321 (-0.248, 0.756)	**-0.395** (-0.641, -0.271)	0
Region: B	**0.434** (0.121, 0.778)	**-0.609** (-0.908, -0.625)	0
Region: C	**0.318** (0.094, 0.595)	**-0.379** (-0.587, -0.355)	0
Region: HN	-0.530 (-1.258, 0.012)	**0.468** (0.408, 0.746)	0
Region: IF	0.260 (-0.048, 0.579)	**-0.347** (-0.445, -0.23)	0
Region: L	-0.287 (-1.042, 0.43)	**-0.714** (-0.99, -0.527)	0
Region: NPC	**-0.570** (-1.064, -0.182)	**-0.185** (-0.376, -0.092)	0
Region: PL	0.388 (-0.009, 0.682)	**-0.462** (-0.66, -0.389)	0
Region: PC	**0.670** (0.293, 1.02)	**-0.525** (-0.755, -0.455)	0
δ	**0.905** (0.896, 0.911)	**0.986** (0.983, 0.992)	**0.968** (0.962, 0.974)
$\hat{\mu}$	**4.350** (4.334, 4.366)	**7.341** (7.172, 7.420)	**6.580** (6.443, 6.688)
$\hat{\sigma}$	**0.183** (0.165, 0.200)	**0.415** (0.076, 0.478)	**1.948** (1.835, 2.023)

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