Canadian Institute of Actuaries
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Educational Note

Calibration of Stochastic Interest rate Models

Document 209122

Most recent version of the document: 221066

This document was archived April 11, 2023

Seeing Beyond Risk
Canadian Institute of Actuaries
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Voir au-delà du risque

Educational Note

Calibration of Stochastic Interest Rate Models

Committee on Life Insurance Financial Reporting

December 2009

Document 209122

Ce document est disponible en français

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Memorandum

To: All Life Insurance Practitioners

From: Tyrone G. Faulds, Chairperson
Practice Council

B. Dale Mathews, Chairperson
Committee on Life Insurance Financial Reporting

Date: December 3, 2009

Subject: **Educational Note – Calibration of Stochastic Interest Rate Models**

The Committee on Life Insurance Financial Reporting (CLIFR), through its Calibration Working Group, has established an initial set of calibration criteria for stochastic interest rate models.

The Working Group has adopted a multi-phase approach in the development of calibration criteria for stochastic interest rate models. The results and recommendations of Phase I of the Working Group's work are contained in the attached educational note.

The Phase I work focused on calibration criteria for long-term, risk-free interest rates, so these will be necessary but not sufficient criteria for models that generate both long- and shorter-term rates. Situations such as the following would be approached with caution:

products backed by short- and medium-term assets,

GICs, short-term annuities with cash out features, and

valuations incorporating credit spreads.

Phase I is directly applicable to Canadian interest rates or instruments denominated in Canadian dollars, but could be adapted for the US and other developed countries with some adjustments.

The next phase will focus on short- and medium-term, risk-free interest rates, and the correlation between short-, medium-, and long-term rates. Future work could focus on credit spreads, other markets, the correlation of interest rates with equities, and the correlation of interest rates with currencies.

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After most of the work was completed on the Phase I calibration criteria, but before this educational note was published, the following two significant events have occurred to warrant further comment,

the current financial crisis, and

publication of calibration criteria by the American Academy of Actuaries.

The financial crisis has produced long-term rates that have, unexpectedly to most observers, been the lowest seen in a half century and there is great uncertainty as to future conditions. We believe that recent events confirm the appropriateness of the calibration approach in areas of

using long history that includes the 1930s and 1940s, and

supplementing historical information with judgment to ensure extremes are reflected appropriately.

No one knows whether the upcoming years and decades will be characterized by sustained economic weakness and low rates, government debt-induced inflation and higher rates, or a return to stability and more moderate interest rate conditions as seen from about 1990 to 2005. However, it is the belief of CLIFR that calibration criteria based upon sufficient history to span all of these possibilities is appropriate. The primary approach to calibrate directly to observed rates, rather than indirectly with a model calibrated to historical rate changes, is generally more robust in this regard.

Recently, we have seen a combination of extremely low rates and high rate volatility that appears unique in modern financial history. These conditions are reflected to a minor extent in the 2-year and 10-year calibration criteria, but fully reflecting these could require volatility varying or regime switching models, which are more complex than the tools used in developing the criteria in this note. The actuary would be cautious if liabilities are sensitive to short-term exposure to high volatility.

In December 2008, the American Academy of Actuaries (AAA) issued a report to the National Association of Insurance Commissioners' Life and Health Actuarial Task Force that addressed calibration criteria. The AAA's work included the publication of stochastic scenario sets, which may be used directly, as well as calibration criteria to be satisfied if actuaries choose to develop their own stochastic models. There are similarities and differences between the AAA and CIA approaches, and the actuary may find it of interest to review the AAA paper.

The focus of this educational note is on calibration of stochastic interest rate models which, generally, will require that a large number of scenarios be run. For valuation purposes, it would not normally be practical to calculate liabilities under each scenario used in the calibration testing, but to use a subset of these scenarios, or a reduced number of scenarios that are meant to represent the full set. Scenario reduction methodologies are beyond the scope of this paper. The actuary may refer to the Standards of Practice and the use of approximations, and other literature that is available that deals with scenario reduction techniques.

Finally, the members of the Working Group have contributed based on their own skills and expertise. The thoughts in the note reflect a general consensus view of the members of the Working Group. Nothing in this educational note would be construed as expressing the views of any of their employers, nor be considered a view or position regarding the policy of the regulators.

In accordance with the Institute's Policy on Due Process for the Approval of Guidance Material Other than Standards of Practice, this educational note has been prepared by CLIFR, and has received approval for distribution by the Practice Council on November 3, 2009.

TGF, BDM

TABLE OF CONTENTS

1. PURPOSE/SUMMARY

The purpose of this educational note is to investigate and develop methodologies and standards for calibration criteria for stochastic interest rate models. The initial focus is on long-term, risk-free interest rates, that is, interest rates with term to maturity of 20 years and longer.

In the Standards of Practice, paragraph 2320.08 makes the following recommendation and paragraph 2330.32 adds the following guidance regarding the selection of stochastic interest rate scenarios:

> 2320.08 *The scenarios of interest rate assumptions should comprise*
>
> *a base scenario, as defined under paragraph 2330.09.1,*
>
> *each of the prescribed scenarios in a deterministic application,*
>
> *ranges which comprehend each of the prescribed scenarios in a stochastic application, and*
>
> *other scenarios appropriate for the circumstances of the insurer.*
>
> 2330.32 If stochastic modelling is performed, the actuary would ensure that the stochastic model includes scenarios that generate policy liabilities outside the range produced by application of the prescribed deterministic scenarios.

Different interest rate models, and parameterizations of models, can produce significantly different sets of scenarios. Notwithstanding any definition for a plausible range on Canadian default-free interest rates, the above provides little guidance on the selection, fitting and use of a stochastic interest rate model. A goal of the Committee on Life Insurance Financial Reporting (CLIFR) was to promote a narrowing of the range of practice, and it was felt that additional guidance would be helpful to the actuary.

It is desirable to have a set of calibration criteria that can be applied consistently to as wide a range of interest-sensitive insurance and investment products as possible, including both long-term and short-term products. Phase I addresses the calibration of long-term, risk-free interest rates, which could then be used in the valuation of products supported by investments in the long-term. The next phase will focus on short- and medium-term, risk-free interest rates, and the correlation between short-, medium-, and long-term rates. Future work could focus on credit spreads, other markets, the correlation of interest rates with equities, and the correlation of interest rates with currencies.

Scenarios produced that satisfy these calibration criteria are not intended to be used to price a bond or derivative.

There are many interest rate models that are available, ranging from fixed to stochastic volatility, and single to multiple regimes. It is not possible to list all of the models, together with the parameters that are required for each. However, general comments and an overview of models are provided in Appendix A.

2. GOALS AND PRINCIPLES

To produce reasonable calibration criteria, the principles below were adopted. The criteria would

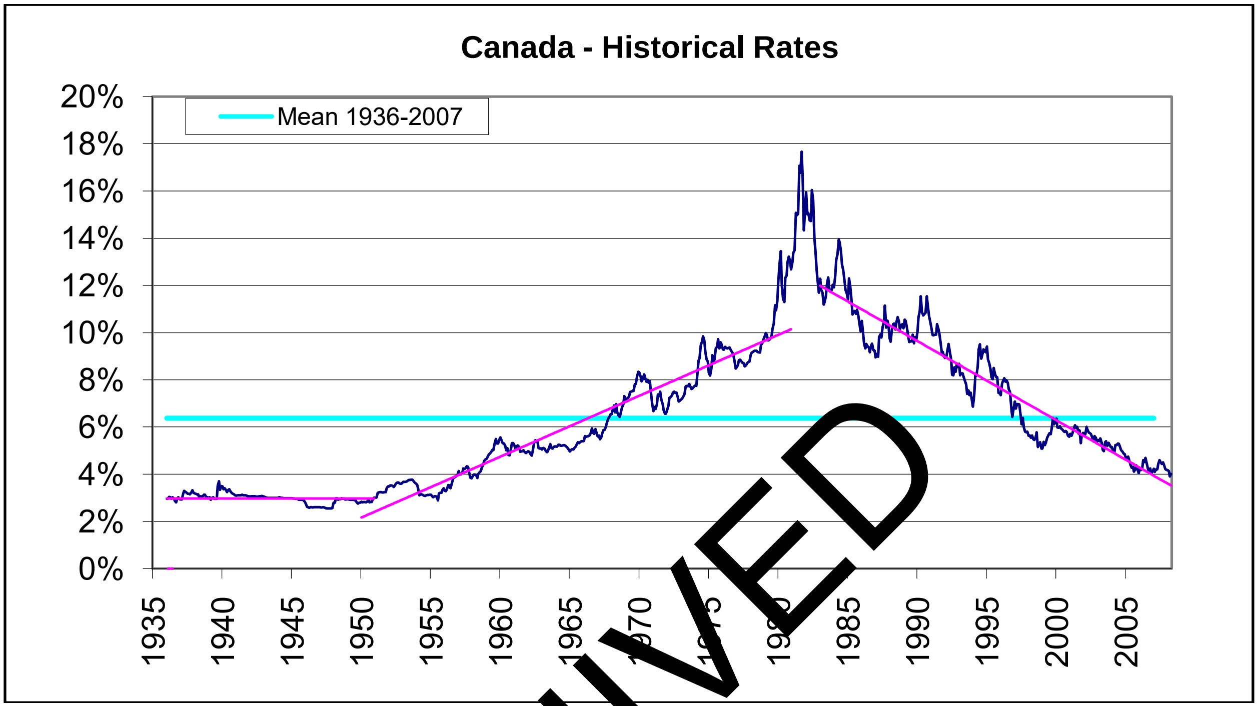
- be sufficiently robust to narrow the range of practice, but allow the actuary to apply reasonable judgment to specific circumstances,
- be applied to the set of scenarios produced,
- be applied to not only the steady state portions of the scenarios produced, but also the near term,
- promote the development of scenario sets that can be used to measure exposure to yield curve shocks as well as long-term paths of declining and rising interest rates, consistent with history, and
- encompass average interest rate distributions corresponding to extended periods of time as well as distributions at selected points in time.

A balance between quantitative and qualitative guidance was considered. A set of criteria based solely on quantitative analysis may place too large a reliance on historical data, can be subjectively influenced by the choice of historical period, and does not take into consideration economic and monetary differences between the historical period and the current time. Qualitative criteria supplement quantitative requirements and encourage the actuary to use economically reasonable models. Qualitative comments are provided for the last two principles above.

Consideration was given as to whether to examine real rates (and inflation) or nominal rates. Nominal rates were chosen since considering the complexity of real rates and inflation was impractical and the availability of historical nominal rates was better. The actuary would refer to the Standards of Practice if inflation consistent with nominal rates is a valuation concern.

3. HISTORICAL INTEREST RATES

Historical long-term Government of Canada bond rates are shown in the following graph:

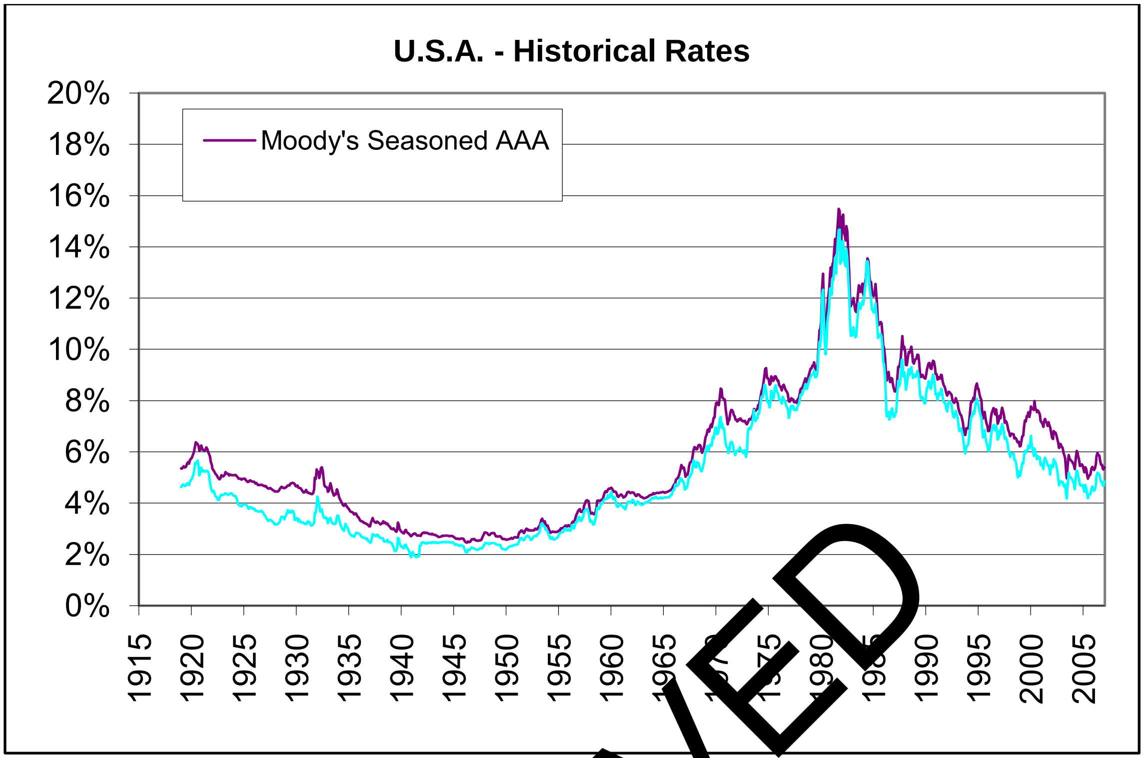


Source: Bank of Canada, Series V122487

From this graph, three distinct patterns can be seen, beginning with the low interest rates of the 1930s depression and through World War II, followed by steadily increasing interest rates through the 1970s and 1980s, and finally a period of steadily decreasing rates to the current day. The dynamics over this 75-year period are complex as there have been changes in the monetary system that may have influenced the level and volatility of interest rates. For example, high inflation and nominal rates were experienced in the 1970s and 1980s, with central banks responding after the 1980s to control inflation.

Within these three periods, interest rates have moved in cycles, consistent with economic cycles. There are short-term cycles, and possibly longer-term cycles. There are periods of sustained low rates, and periods of extreme highs that tend to be of short duration.

Historical US 30-year Constant Maturities Treasuries and Seasoned AAA corporate bond rates are shown in the graph below. The US interest rates show similar patterns to those in Canada.



Source: Federal Reserve Bank of St. Louis

Based on these data, a well-designed model would have the following characteristics. It would

produce a wide range of outcomes, consistent with historical ranges,

produce periods of sustained lows,

produce periods of sustained highs (but with low probability of sustained extreme highs),

produce periods of trending low or trending high rates, and

move between lows and highs over reasonable periods of time.

4. CALIBRATION CRITERIA FOR LONG-TERM INTEREST RATE MODELS

This section gives an overview of the complete set of calibration criteria for long-term, risk-free interest rates. The sections that follow describe each component of the calibration criteria in more detail.

The methodology used to develop the calibration criteria reflected a desire to be guided by history but also to apply judgment considering that the past, at best, can give only an indication of how the future may look. It is appropriate to assume that economies and financial markets move in cycles (and/or irregular fluctuations), and that there will likely be extremes and periods of high rates and periods of low rates. In this sense, a multi-faceted view was taken by looking at history, by discussing and considering what is different today versus the past, and by discussing what may be different in the future.

The normal approach for building a model and generating interest rate scenarios would be to choose a model and then select parameters using an appropriate procedure. The scenario results from the model would then be examined to determine if calibration criteria were satisfied. If necessary, the parameters would then be adjusted in order to produce revised interest scenarios that satisfy the calibration criteria.

Calibration consists of the three requirements,

> satisfying calibration criteria in each of the left and right tails of the distribution,
>
> producing a reasonable median rate, and
>
> satisfying a mean reversion constraint.

Calibration criteria have been developed for the 2-year, 10-year, and 60-year horizon points of the projection. Interest rate scenarios at the 2-year and 10-year horizons will be influenced by the initial starting interest rate, so calibration criteria at each of a 4%, 6.25%, and 9% starting interest rate are provided. At the 60-year horizon, the impact of the starting rate would be expected to be minimal, so only calibration criteria at a starting rate of 6.25% are provided. Models will generally be constrained by a subset or a few key percentiles at each of the measurement points. For example, of the six criteria at the 60-year horizon point, a model is typically constrained by only two. However, these key points will vary from model to model.

The calibration criteria are focused on the tails of the distribution (i.e., ≤10th percentile and ≥90th percentile) as follows:

Calibration Criteria for Long-Term Interest Rates

		2-Year Horizon			10-Year Horizon			60-Year Horizon
		Initial Rate			Initial Rate			Initial Rate
		4%	6.25%	9%	4%	6.25%	9%	6.25%
Left-tail Percentile	2.5	2.95%	4.40%	6.20%	2.50%	3.20%	4.00%	2.60%
	5	3.10%	4.65%	6.55%	2.70%	3.50%	4.45%	2.95%
	10	3.30%	4.95%	6.95%	3.00%	3.90%	5.00%	3.40%
Right-tail Percentile	90	5.05%	7.70%	10.70%	6.60%	9.05%	11.60%	10.00%
	95	5.40%	8.15%	11.30%	7.45%	10.25%	12.80%	12.00%
	97.5	5.70%	8.60%	11.80%	8.25%	11.40%	13.90%	13.50%

These criteria will be satisfied if the model produces percentile values that are less than or equal to each of the left-tail criteria, and greater than or equal to each of the right-tail criteria.

In addition, at the 60-year horizon, a median outside a 5.00% to 6.75% range would generally be considered unreasonable. The decision to select a model with a median outside this range would need to be supported by a clearly documented rationale.

Finally, for all models, the rate of mean reversion would not be stronger than 14.5 years (equivalent to a half-life of 10 years).

Models calibrated to these criteria will produce long-term, risk-free interest rate scenarios that demonstrate both rapidly rising and falling long-term rates for both high rate and low rate environments (i.e., shocks), and patterns of gradually declining and increasing interest rates. The mean reversion criteria will also produce scenarios that show periods of sustained highs and lows. All of these features are desirable and consistent with historical experience.

The sections below describe the development of the calibration criteria in more detail.

5. STEADY STATE CALIBRATION CRITERIA

The “steady state” is defined to be the point in time beyond which the distribution of model generated interest rates changes only negligibly. This point can be very far in the future. For practical reasons, the calibration criteria are defined at a specific point in time of 60 years.

Calibration started with the steady state because, by definition, it is independent of the starting yield environment. From the steady state, calibration criteria at nearer points in the projection horizon were defined, where the starting yield environment does influence outcomes.

The chart below provides the long horizon calibration criteria that would be met by the model,

Steady State Calibration Criteria		
Percentile		Rate
Left-tail	2.5th	2.60%
	5th	2.95%
	10th	3.40%
Right-tail	90th	10.00%
	95th	12.00%
	97.5th	13.50%

These criteria will be satisfied if the model produces percentile values that are less than or equal to each of the left-tail criteria, and greater than or equal to each of the right-tail criteria.

Rather than specific quantitative calibration criteria for either the mean or the median of the distribution, more general guidance has been provided. From 1936 to 2007, Canadian risk-free long bonds had mean and median returns of 6.35% and 5.55%, respectively.

The 40th to 60th percentiles are 5.08% and 6.72%, respectively. Models producing a median value lower than the mean would be consistent with this history. A range of values around the historical median would be acceptable, although, a median outside a 5.00% to 6.75% range would be considered unreasonable, in the absence of justification.

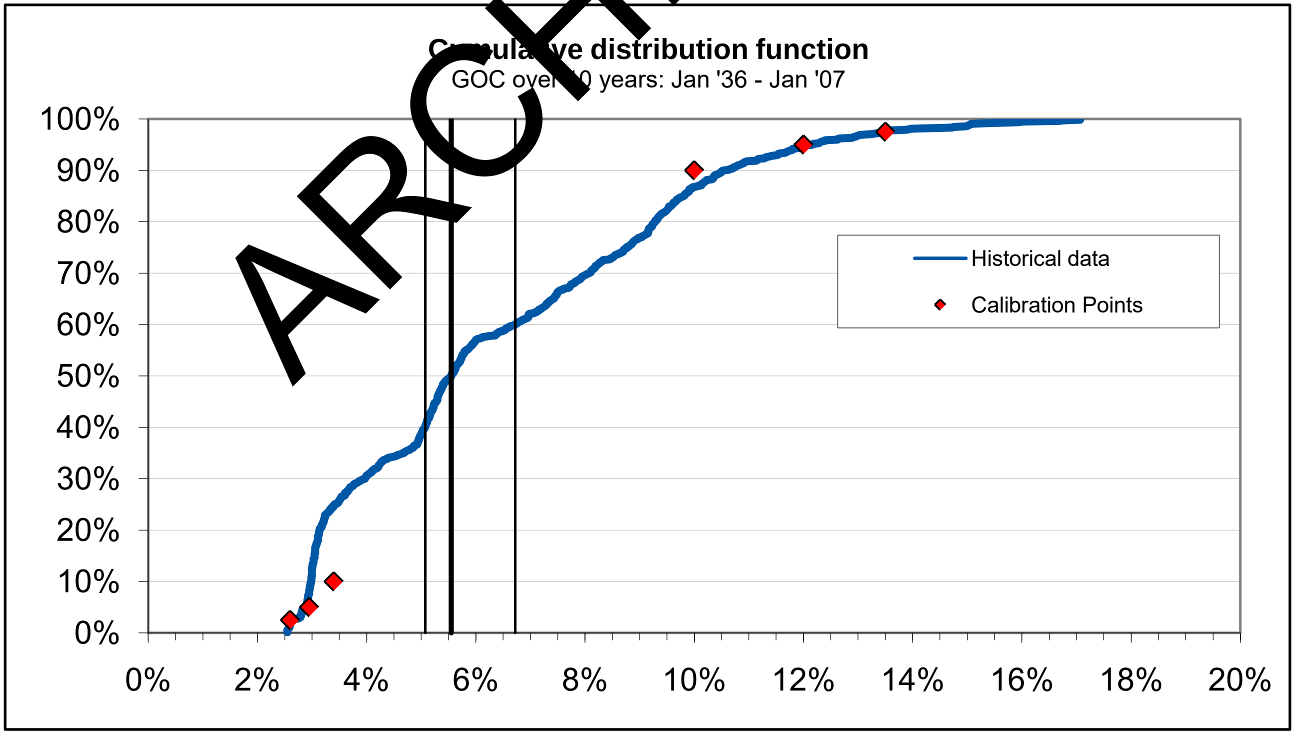
5.1 Comparison to Historical

The following table and graph show that the calibration criteria are consistent with history at most calibration points.

Comparison of Steady State Calibration Criteria to Historical Interest Rates

Percentile		Rate	1936-2007	Difference
Left-tail	2.5th	2.60%	2.61%	(0.01)%
	5th	2.95%	2.90%	0.05%
	10th	3.40%	2.99%	0.41%
Right-tail	90th	10.00%	10.56%	(0.56)%
	95th	12.00%	12.16%	(0.16)%
	97.5th	13.50%	13.44%	0.06%

The shapes of cumulative distribution functions that models would reasonably be able to produce were considered. Requiring a closer fit to history would be over conservative and difficult to achieve when also considering the differences between the economic and financial environments of historical periods compared to recent experience.



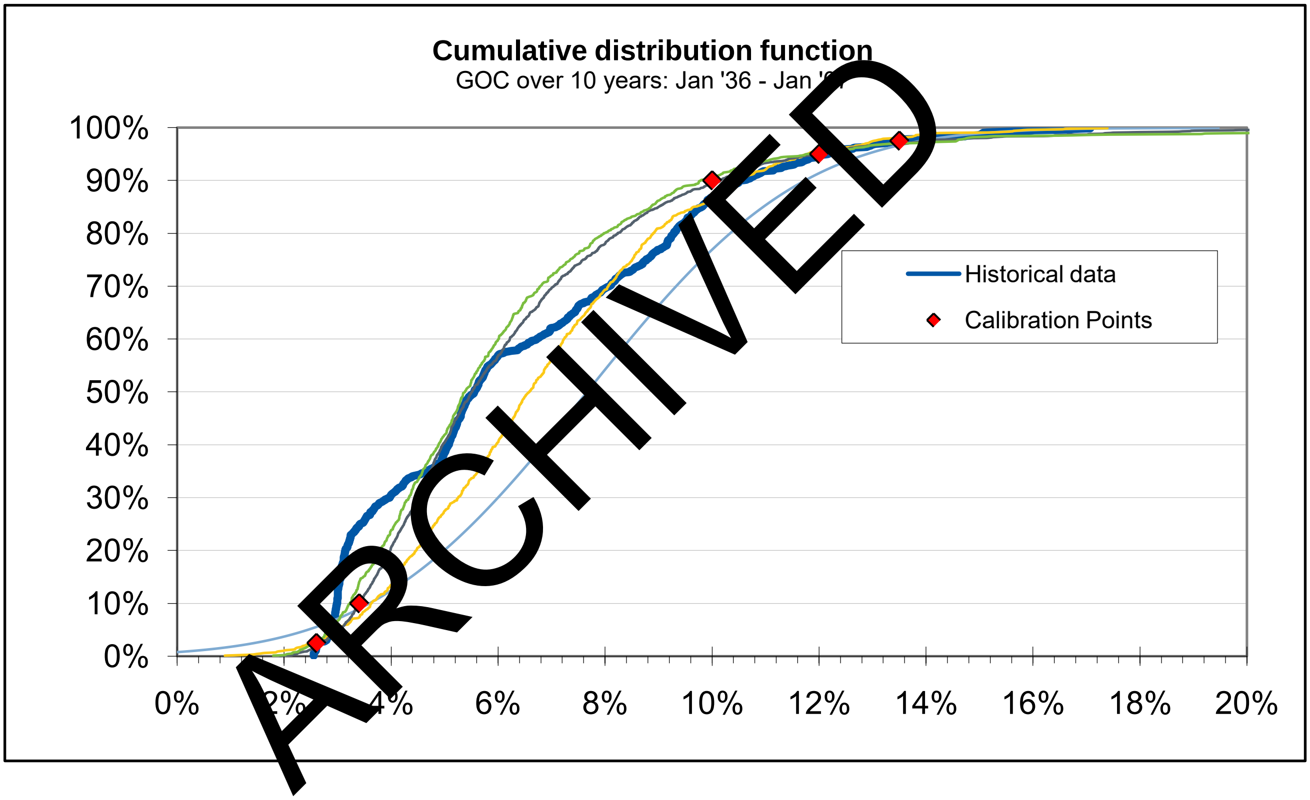
Source: Bank of Canada, Series V122487

5.2 Model Testing Results

The steady state calibration was tested against several commonly used and publicly available models. The aim of the model testing was to determine whether common model forms with reasonable parameterizations could produce scenarios that satisfied the calibration criteria.

This was accomplished by testing different types of models, including Vasicek (VAS), Cox-Ingersoll-Ross (CIR), Brennan-Schwartz (BS), and Multiplicative Shock (MS). The CIR, BS and MS models were calibrated assuming a 15-year reversion period. Details on the CIR and BS model setup are provided in Appendix B.

Results from some of the testing are shown in the following graph,



Source for historical data: Bank of Canada, Series V122487

The testing shows that the VAS and CIR models are constrained by the 10th and 97.5th percentiles while the BS and MS models are constrained by the 2.5th and 90th percentiles. This is also apparent in the following tables of model point results.

Steady State Calibration Criteria - Model Testing Results

Percentile	Criteria	Vasicek	CIR	BS	MS
1st		0.14%	1.83%	2.30%	2.30%
2nd		0.99%	2.17%	2.52%	2.52%
2.5th	2.60%	1.29%	2.30%	**2.60%**	**2.60%**
5th	2.95%	2.27%	2.78%	2.90%	2.91%
10th	3.40%	**3.40%**	**3.40%**	3.28%	3.29%
median		7.39%	6.34%	5.47%	5.47%
90th	10.00%	11.39%	10.58%	**10.00%**	**10.00%**
95th	12.00%	12.52%	12.07%	12.18%	12.12%
97.5th	13.50%	**13.50%**	**13.53%**	14.63%	14.57%
98th		13.79%	13.98%	15.55%	15.45%
99th		14.64%	15.45%	18.59%	18.50%

In addition, the model results show a range for the 50th percentile (median), with VAS as an outlier. The 40th to 60th percentile range of historical results has been 5.08% to 6.72%. The median rates for CIR, BS and MS models are within this range. The Vasicek result lies outside this range, and would not satisfy the criterion discussed in section 5.

Testing has focused on single regime models, with only limited work conducted on a multiple regime model. The calibration criteria are equally applicable to fixed volatility and stochastic volatility models. However, the actuary using stochastic volatility models would consider carefully that any model satisfying the calibration criteria is parameterized in a reasonable way and makes economic sense.

5.3 Starting Rate for Calibration

The initial approach was to use the current yield curve as the initial starting point to test the calibration criteria, under the assumption that the steady state distribution is independent of the starting rate. Testing showed that when a relatively weak mean reversion is used in the selected models, the initial starting rate effect persists for a very long time.

To avoid having to calibrate the model frequently for small changes in interest rates over short periods of time, it is recommended that the model be calibrated using an initial rate of 6.25% and a projection horizon of 60 years. The actuary would test and ensure that sufficient scenarios are run such that the steady state distribution is stable.

This approach also addresses the practical issue that, in most cases, models will be parameterized and tested, and scenarios run, in advance of the valuation date, and it is likely that interest rates will change over this period.

6. 2-YEAR AND 10-YEAR CALIBRATION CRITERIA

For calibration at shorter time horizon points, the initial starting rate is important. For this reason, criteria suitable for low, average and high interest rates at the starting environment were developed. A calibrated model would satisfy the steady state criteria, and also the criteria for each of the following three starting interest rates.

2-Year and 10-Year Calibration Criteria							
		2-Year Horizon			10-Year Horizon		
		Initial Rate			Initial Rate		
		4%	6.25%	9%	4%	6.25%	9%
Left-tail Percentile	2.5th	2.95%	4.40%	6.20%	2.50%	3.20%	4.00%
	5th	3.10%	4.65%	6.55%	2.70%	3.50%	4.45%
	10th	3.30%	4.95%	6.95%	3.00%	3.90%	5.00%
Right-tail Percentile	90th	5.05%	7.70%	10.70%	6.60%	9.05%	11.60%
	95th	5.40%	8.15%	11.30%	7.45%	10.25%	12.80%
	97.5th	5.70%	8.60%	11.80%	8.25%	11.40%	13.90%

To determine these criteria, historical results were initially reviewed. However, since limited data are available to analyze the progression of rates from each of these starting rate environments, results from the CIR and BS model forms that had been used to test calibration of the steady state were mainly utilized to develop the shorter horizon criteria, with some judgment applied. To set the 2-year and 10-year calibration criteria, the results from the model that produced a narrower dispersion of interest rates were chosen. Models that satisfy these criteria will produce a reasonable spread of results at both 2-year and 10-year horizons.

It is likely that models calibrated to satisfy the steady state criteria will also meet the 2-year and 10-year criteria above with few, or no, further adjustments to parameters. It is reasonable to have these additional criteria at 4% and 9% starting rates, because history has shown that interest rates can move significantly over short periods of time.

7. MEAN REVERSION CRITERIA

Historical experience has shown that interest rates can stay at low levels for extended periods of time. The calibration criteria designed up to this point do not sufficiently constrain models to reflect economic environments where interest rates remain at low levels over an extended number of years.

For this reason, an additional constraint was necessary for all models so that the rate of mean reversion would not be stronger (i.e., not shorter or quicker) than 14.5 years (which is equivalent to a ½ life of 10 years).

For simple models with an explicit mean reversion factor, this requirement can be satisfied by considering the value of the mean reversion parameter directly. For more complicated models, this requirement can be satisfied by using a mathematical proof or using the procedure in Appendix C.

8. SCENARIO GENERATION

Once all the calibration criteria have been satisfied, and using the actual starting interest rate, the actuary would run the interest rate scenarios using the same model, parameters, number of scenarios, and random number seed[1] as was used to test the calibration. The interest scenarios may then be used for valuation purposes or other work. It is possible that only a subset of the scenarios would be used. A discussion on scenario reduction techniques is beyond the scope of this educational note, and the actuary would consult the literature that is available on this subject. The actuary may also refer to subsection 1510 of the Standards of Practice on the use of approximations.

9. CALIBRATION CRITERIA FOR OTHER COUNTRIES

The scenarios produced from models that satisfy the calibration criteria would be appropriate for valuations utilizing long-term, risk-free reinvestment assumptions. While the criteria have been developed utilizing Canadian data, they could be applied to US government bonds and many (but not all) other developed economies. An actuary building a model for these non-Canadian economies would consider these criteria as a starting point and make adjustments as he or she judges appropriate. In making such a judgment, rate history, market information, economic and political conditions may be considered. If calibration criteria relevant to the particular country or currency being modeled have been published, they could be used as an additional source of information and guide to aid the actuary in forming his or her opinion. It may be acceptable to use those criteria if it can be demonstrated that they are broadly consistent with the criteria in this educational note (either the criteria themselves are broadly consistent, or the approach taken to develop the criteria is broadly consistent with this educational note). In the absence of such a demonstration, it would not be appropriate to utilize the other country's criteria.

Countries with recent or extended histories of either unusually low rates or high rates would be examples where the criteria may not be appropriate. In some countries, history may be limited, and a wider distribution of rates relative to these limited observations may be needed in order to provide a margin for uncertainty.

Finally, the calibration criteria would not be appropriate for developing and emerging markets.

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[1] The actuary would verify that the corresponding numbered scenarios behave in a similar manner and converge in a similar way, to ensure there has not been an inadvertent "reshuffling of the random stochastic deck" upon shifting to the current interest environment for the interest rate scenarios.

APPENDIX A

The CALM liability is determined by modeling the asset and liability cash flows over a defined set of scenarios, and comparing the resulting liability balances. If the deterministic approach is taken, the set of scenarios are the ones prescribed in subsection 2330 of the Standards of Practice plus whatever supplemental scenarios the actuary deems appropriate to the risk profile of the liabilities. The liability is set to be in the upper part of the resulting range of policy liabilities, and at least as great as the highest policy liability resulting from the prescribed scenarios. If a stochastic approach is used, a large number of different interest rate scenarios are generated randomly, with the liability calculated under each scenario. The liability is set (at the discretion of the actuary) to be between the CTE 60[2] and CTE 80 result.

Stochastic Modeling

The stochastic modeling of interest rates is similar to the stochastic modeling of equity returns (which is in general use to support liabilities for annuity investment guarantees). It differs in that an important part of the modeling of interest rate movements is an assumption of non-negative rates, and generally some form of reversion to a mean. The mean is usually chosen with regard to a relevant body of historical interest rates. The model used will define how rates move from one period to the next through a formula applied to values generated through a Monte Carlo simulation. The parameters in the model represent mean, volatility, and usually the strength of the reversion to the long-term mean. Our current note on calibration criteria does not prescribe the model form, or the setting of the parameters, but rather focuses on the scenarios resulting from an application of the scenario generator. This allows the actuary flexibility in the selection of one of the standard model formulations, or the modification of one of the standard formulations to create a new model that provides a better fit for the individual application under analysis.

Choice of Stochastic Modeling over Deterministic Modeling

Stochastic modeling is not a radical departure from deterministic measures. It is an enhanced form of scenario testing whereby a wide range of random scenarios are developed using a model that is a representation of real life. In deciding whether stochastic modeling of interest rates would be utilized for the valuation, the actuary would consider the complexity of the interaction of interest rates with the asset and liability cash flows within the CALM model, as well as the materiality of the impact of the interest rate volatility on results. If the product design is such that most of the liability outflows will occur within a relatively narrow range around the mean of the distribution of outcomes, an approach of using the best estimate plus an explicit margin is appropriate. If, however, there are high benefit outflows that only happen in low probability areas of the distribution (the tails) then a stochastic approach can give a more appropriate picture of the extent of interest rate exposures. Stochastic modeling may also be the preferred approach where there is no natural best estimate, such as when modeling interest rates that will be available for reinvestments 25 years or more in the future.

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[2] Conditional Tail Expectation (CTE). CTE 60 is the average of the 40% highest liabilities resulting from the application of all of the scenarios in the set.

Practical Considerations

The stochastic CALM liability is set as the average of a subset of the highest resulting liabilities. It is important to note that this can mean that the liability is an average of scenarios that are neither the lowest interest rate scenarios nor the highest rate scenarios. For example, consider a product with high net positive cash flows from premiums in the next ten years, and negative cash flows emerging over the subsequent ten years, so that by year 20 the bulk of the cash flow is negative as benefits outweigh premiums and asset cash flows. An adverse scenario here will feature low interest rates in the first ten years and higher rates in the years past year 20. This is a natural outcome of the stochastic modeling. If there is a need to develop a single average interest rate vector for the purpose of subdividing a block of business after the CALM run, then an odd pattern is possible.

Sample Types of Models in Common Use

In this section simple interest rate model examples are provided. The goal of this section is to illustrate how some models can be calibrated and used. This would not be construed as a recommendation of a specific model. As a matter of fact, more complex models will allow a better adjustment to the calibration points.

A common characteristic of the models considered here are that they are mean-reverting models. Mean reversion is a recognized property of interest rates, which is well documented in the available financial literature.

The models presented here are used to model long-term interest rates. They are characterized by a drift process and a stochastic process. The drift process defines the mean-reverting rate and the speed of reversion. The stochastic process varies among the models presented here. The variance scales the volatility in terms of the current interest rate level.

Vasicek Model Form

References to a model form as the model is applied to project the long-term rate. The original Vasicek model is a model of short-term interest rates. The model, in its discrete form, is given by the equation.

$$r_t = (1-\alpha) r_{t-1} + \alpha\tau + Z_t \, ,$$

where $Z_t \sim N(0, \sigma^2)$.

This model requires three parameters.

τ is the long-term steady-state rate to which the process is reverting,

α is the strength of mean-reversion, must be between 0 and 1, a zero value would result in no mean-reversion while a one value would result in full reversion in next period, and

σ is the volatility parameter of the stochastic process.

To determine the parameters for this model from a set of historical data, such as the series V122487 from CANSIM representing the Government of Canada over 10-year bond yields, a constrained Maximum Likelihood Estimation (MLE) approach can be used.

For the Vasicek model form, which is an autoregressive model, the projected steady-state interest rate will follow a normal distribution given by

$$r_t \sim N\left(\tau, \frac{\sigma^2}{1-\alpha^2}\right).$$

This normal distribution of rates implied by the model results in the situation that it is almost impossible to avoid negative interest rate scenarios while producing scenarios with rates that reach levels observed in the early 1980s.

Because we have nine calibration points while the model has only three parameters, and the fact because of that the steady state rate follows a normal distribution, it is possible to develop a closed form solution for the parameters that will exactly meet the key calibration points (10th and the 97.5th percentiles). Other calibration points are not likely to be a constraint for the Vasicek model form.

Cox-Ingersoll-Ross (CIR) Model Form

This model is similar to the Vasicek model, but differs in that the stochastic process is scaled by the square root of the interest rate. This ensures that the interest rate does not become negative, because the closer the interest rate r_t gets toward zero, the closer the formulation becomes to a mean reversion process with no stochastic term. This model is represented by

$$r_t = (1-\alpha)r_{t-1} + \alpha\tau + \sqrt{r_{t-1}}\,Z_t,$$

where $Z_t \sim N(0, \sigma^2)$

Like the Vasicek model form, the CIR model form also requires three parameters that have the same interpretation.

Again, it is possible to determine the parameters for this model using a set of historical data, with a constrained MLE approach. However, in this case, the steady state rate does not follow a normal distribution. Simulations can be used to calculate the needed percentiles. If such an approach is used, then it is necessary to fix a reasonable duration at which we presume that the steady-state rate is achieved as the distribution continues to expand over time.

The advantage of this model is that it produces a skewed distribution. It is possible to generate scenarios with rates reaching levels observed in the early 1980s and avoid negative interest rates in other scenarios.

Because we have more calibration points than model parameters, it is also possible to develop a set of parameters that will exactly meet the calibration points for a given duration.

Brennan-Schwartz Model Form

This model is similar to the CIR model, with the difference that the stochastic process is now scaled by the interest rate. This model is represented by

$$r_t = (1-\alpha)r_{t-1} + \alpha\tau + r_{t-1}Z_t \, ,$$

where $Z_t \sim N(0, \sigma^2)$.

In this case, due to the stronger scaling of the volatility parameter, the distribution of the long-term interest rate will present more skew. This characteristic of the model provides a better fit to historical data than the previous models.

Multiplicative Shock Model

This model has a different formulation as its stochastic component follows a log-normal process. This model is represented by

$$r_t = [(1-\alpha)r_{t-1} + \alpha\tau]e^{Z_t} \, ,$$

where $Z_t \sim N\left(-\tfrac{1}{2}\sigma^2, \sigma^2\right)$.

Interestingly, the distribution of the long-term interest rates generated by this model is very similar to the Brennan-Schwartz model form.

APPENDIX B

This appendix provides a sample of two models and parameter sets used in the testing and development of the calibration criteria.

This information is provided for transparency and to assist the actuary in understanding how the models work and are calibrated. The actuary is cautioned against simply utilizing these models in his or her work, but to develop sufficient expertise to apply actuarial judgment in stochastic model form and parameter selection, consistent with the calibration criteria.

The following form of the **Brennan-Schwartz** (volatility parameter as power of rate = 1) model was used

$$r_t = (1-\alpha)r_{t-1} + \alpha\tau + r_{t-1}Z_t \,,$$

where $Z_t \sim N(0,\sigma^2)$.

long-term mean = 6.23%

mean reversion weight to equilibrium rate .00291 (or .99709 weight to prior observation, equivalent to 28.6 years period = 1/(12*.0291)

volatility = .03524 (multiply by 12^.5 to annualize), multiply by rate to translate units from fraction of rates to rates themselves

number of scenarios generated = 50,000 (a trinomial tree application was also used).

The following form of the **Cox-Ingersoll-Ross** (volatility parameter as power of rate = 0.5) model was used

$$r_t = (1-\alpha)r_{t-1} + \alpha\tau + \sqrt{r_{t-1}}Z_t \,,$$

where $Z_t \sim N(0,\sigma^2)$.

long-term mean = 6.77%

mean reversion weight to equilibrium rate .00440 (or .99560 weight to prior observation, equivalent to 18.9 years period = 1/(12*.0440)

volatility = .01046 (multiply by 12^.5 to annualize), multiply by rate^.5 to translate units from fraction of rates^.5 to rates themselves

number of scenarios generated = 50,000 (a trinomial tree application was also used).

APPENDIX C

One purpose of the calibration criteria is to ensure that scenarios robustly represent periods of sustained low rates, which limit investment income on reinvestments needed to support long-term guarantees. Although single point in time tail criteria go some way to ensuring this outcome, they do not exclude models that produce scenarios in which periods of low rates tend not to be sustained, so that few scenarios would display low interest rates averaged over a potentially extended period during which reinvestment could be financially important. Sustained periods of low rates can be statistically demonstrated if the scenarios that are relatively low in early years tend to stay relatively low in later years. As an example, although other approaches are possible, and as an alternative to a mathematical proof, satisfaction of this criterion can be demonstrated with the following procedure.

1. Sort Scenarios for lowest to highest long-term rate at projection year T0, where T0 is sufficiently long to accumulate substantial dispersion in rates, but not so long as to be beyond most expected reinvestments. For a typical long-term guaranteed block, T0 might be in the range of 5 to 10 years.
2. Group the scenarios by rate quartile at T0, from lowest (Quartile 1) to highest (Quartile 4). Calculate the magnitude of dispersion of low rate scenarios from central scenarios dispersion (T0) = Average rate (T0) within Quartile 1 – average rate (T0) within combined (Quartile 2 & Quartile 3).
3. Using the same scenario grouping (ranked at T0, **not** re-ranked at T0+10) calculate 10 year later dispersion (T0+10, ranked T0) = Average rate (T0+10) within Quartile 1 – average rate (T0+10) within combined Quartile 2 & Quartile 3.
4. The mean reversion criterion over the projection period from T0 to T0 +10 is satisfied if dispersion (T0+10, ranked T0) > = 0.5 * dispersion (T0).
5. If the actuary can demonstrate that the model rate of mean reversion is similarly robust across other projection periods, this single test would be sufficient. If not, the test would be repeated across sufficient financially meaningful periods to demonstrate sustained periods of low rates.
6. If periods of sustained high rates are financially stressful for a particular application in the opinion of the actuary, the demonstration would be repeated for these rates (Quartile 4 relative to quartiles 2 & 3).

A model with a single regime and simple linear mean reversion (i.e., E(r(t+dt) =r(t) + (1/ reversion period)* dt* (long-term mean – r(t)) can be demonstrated to satisfy this criteria (with sufficient numbers of scenarios) if the reversion period > 14.5 years[3]. If the projection period (dt) is greater than one month, the mean reversion period threshold may need to be slightly adjusted.

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[3] With this simple mean reversion, at the continuous limit, E(r(t+n))=long-term mean +exp(-n/reversion period) *(r(t) –long-term mean). For an elapsed period n of 10 years, the exponentially decaying weight on initial rate will be >= 0.5 when mean reversion period >= 10/ ln(2) =14.42.

Models would generally not be used with characteristics that would invalidate the statistical intent of this criterion (i.e., a cyclical component of rates with roughly 10-year periodicity). If exceptional circumstances make such a model appropriate in the opinion of the actuary, the actuary would develop robust statistical methods appropriate to the model characteristics to demonstrate substantive sustained periods of low rates, consistent with this criterion.

Finally, it appears likely that models that satisfy both the long-term equilibrium tail criteria, and reproduce close to historically representative volatility, will also satisfy this mean reversion criterion, although some models may possibly require modest parameter adjustment. Some work has been done that suggests that the criteria used here are consistent with statistically plausible long-term rates of mean reversion. Some mean reversion estimates based upon statistical fit to rate change history may estimate somewhat stronger (shorter period) or weaker (longer period) mean reversion than that of this criteria. Statistical estimates of mean reversion tend to have large uncertainty, and may vary greatly depending upon the specific historical period used for estimation. Therefore, mean reversion that is stronger than that of this criterion, even if it is a statistical best estimate, may provide spurious comfort regarding the potential likelihood of sustained periods of extreme rates.