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# High-Water Mark Fee Structure in Variable Annuities

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## Abstract

The fee structure of variable annuities is important for both insurers and policyholders. A well-designed fee structure may help reduce the risk exposure of the insurer and simultaneously increase the welfare of the policyholder. In light of this, we hereby propose a variable annuity with a novel high-water mark fee structure, and examine its implications for both the insurer and the policyholder.

From the insurer's perspective, we first determine the fair insurance fees within the conventional risk-neutral pricing framework and later discuss the insurer's risk management implications.

From the policyholder's perspective, we consider three types of policyholders with a mean-variance preference, namely a naive policyholder, a naive policyholder with exogenous shocks, and a sophisticated policyholder. We first evaluate these policyholders' welfare in the context of the variable annuity with the high-water mark fee structure. Later, a comparative analysis of policyholders' welfare under a constant and a state-dependent fee structure is also included. We find that the high-water mark fee structure can generally increase the policyholder's welfare in comparison to the other two fee structures. Specifically, the high-water mark fee structure is more favourable for policyholders who are more risk-averse, more likely to experience exogenous shocks, and have a shorter contract length. Also, we observe that the high-water mark fee structure is more robust in policyholders' welfare across a range of risk preferences and is therefore more marketable than the other two fee structures.

## 1 Introduction

Variable annuities (VAs) are equity-linked insurance products issued by insurance companies. Their flexible investment options, favourable tax-deferral treatment, and stable long-term guarantees have made them one of the most prevalent investment vehicles over the last two decades. At the inception of a VA contract, a policyholder (PH) pays a lump sum initial premium to an insurer, who invests the sum into a basket of preassigned mutual funds (often referred to as the policy fund) by setting up an investment account to track the performance of the policy fund. Payouts under the VA policy are often subject to some minimum guarantees which kick in when the performance of the policy fund is poor. To fund these guarantees, the insurer periodically depletes the investment account by charging insurance fees. The PH is also given the option to surrender the VA contract before maturity subject to some predetermined surrender penalty.

In light of the above, a standard VA policy with minimum guaranteed payouts offers the PH protection against bearish market conditions while allowing the PH to financially gain from bullish market movements. On the other hand, the insurer has exposure to a variety of risks, including financial market risk, mortality risk, PH behaviour risk (e.g., surrender risk), and others.

Nevertheless, even with the seemingly great advantages to PHs, the VA market has experienced dwindling sales over the past half-decade. Many reasons have been evoked to explain this trend, most notably the high insurance fees associated with VA products (see Bernard and Moenig, 2018). The sluggish VA market highlights the importance of the fee structure design in the marketability of VAs. On the one hand, different fee structures induce different incentives for PHs whose preferences and objectives are somewhat unique to the individual level. On the other hand, by affecting PHs' behaviour, each fee structure has its own unique implications on an insurer's risk management activities. A preferable fee structure is one balancing the interests and preferences of both insurers and PHs.

For VA products, the most prevalent fee structure is the so-called constant fee structure, which termly levies a fixed percentage of the investment account as insurance fees. As highlighted by many researchers, this time- and state-invariant fee structure is known to incentivize PHs to surrender the VA contract when the investment account grows in value as PHs pay a high insurance fee for guarantees with negligible values. In view of this problem, a few research papers have looked into insightful solutions to design more favourable fee structures. For instance, Bernard and Moenig (2018) propose a time-dependent fee structure where the insurance fee is reduced after a certain time threshold. By implicitly discouraging PHs' surrender behaviour, the time-dependent fee structure is shown to reduce insurance fees while keeping the VA contract profitable to insurers. Bernard et al. (2016) and Cui et al. (2017) consider a VIX-linked fee structure and show its appeal to help re-align the collection of insurance fees with the insurer's liability, which might in return help reduce the PH's surrender incentive. Most relevant to the present work, Bernard et al. (2014) introduce a state-dependent fee structure that charges a constant fee only when the value of the investment account is below a certain threshold. It is later shown by MacKay et al. (2017) that the state-dependent fee structure can make the surrender behaviour completely sub-optimal by imposing a certain marketable surrender penalty.

Although most of the aforementioned papers show that the newly proposed fee structures are helpful to better align the insurer's liability with the collection of insurance fees (consequentially beneficial for the insurer in reducing the surrender risk), little is known about the impact these fee structures may have on PHs. The main drawback of the constant fee structure is that the fixed percentage fee fails to adjust to PHs' incentives under various market conditions and therefore undermines PHs' welfare, which eventually erodes the marketability of VAs. However, by mainly focusing on the insurer's risk management implications, the analysis fails to comprehensively consider the interests of both parties in the financial transaction.

In light of the above, the goal of this paper is to propose a VA with a novel *high-water mark* (HWM) fee structure and examine its merits for both the PH and the insurer. This HWM fee structure is assumed to have a blend of features from a state-dependent constant fee and a pure HWM fee. More specifically, in addition to a state-dependent constant fee, the HWM fee structure is designed to charge a HWM fee when the investment account reaches new record highs above a certain threshold. The HWM fee structure will be shown to reduce the variance of the VA payouts by stabilizing the investment account, a desirable feature for a risk-averse PH. This is attributed to the inherent design feature of the HWM fee structure of charging a reduced constant fee when the investment account value is low while exercising some restraints on the growth rate of the investment account when the financial market is performing well. Through the specification of the HWM fee, the resulting VA will be shown to display varying levels of stability. However, this will come at the expense of aggravating the surrender risk when the investment account becomes large in comparison to the state-dependent fee structure.

Note that the HWM fee is frequently applied in the hedge fund industry where a Two-and-Twenty fee scheme[1] is widely accepted. The impact of the HWM fee on investors and fund managers in hedge funds has been well documented; see, for example, Guasoni and Obłój (2016) and references therein. However, we would like to point out that the proposed application of a HWM fee in VAs is fundamentally different to its use in hedge funds. Indeed, in hedge funds the HWM fee is used to compensate managers for their skills, while in VAs it is utilized to fund the embedded guarantees.

As for the analysis, we propose to adopt the risk-neutral pricing approach to determine the fair insurance fees[2] for the VA with the HWM fee structure. This is the conventional approach to price VAs in the literature; see, for example, Bauer et al. (2008), Dai et al. (2008), Huang and Kwok (2016), Milevsky and Salisbury (2001, 2006), Milevsky and Salisbury (2002), and references therein.

Within the risk-neutral pricing framework, it is opportune for PHs to maximize the expected present value (EPV) of future cash payouts since each agent is able to replicate every possible cash payout in a complete market with no friction. In what follows, we refer to this optimal strategy of PHs as the *risk-neutral-pricing* strategy. Assuming that PHs follow the risk-neutral-pricing strategy is important for pricing, although this strategy might deviate from empirical PH behaviour. The reason is that this strategy corresponds to the worst-case liabilities (also referred to as the *hedging cost* in Forsyth and Vetzal, 2014) for insurers. In other words, by assuming this strategy when pricing, insurers can hedge every possible cash outflow with no risk.

From the PH's perspective, we evaluate their welfare in holding the VA assuming a mean-variance (MV) risk preference. We carry out a comparative analysis by also considering the PH's welfare under the constant and state-dependent fee structures. In a complete and frictionless market, the risk-neutral-pricing strategy should fully capture PHs' behaviour (see Bauer et al., 2017, for details). However, the life insurance market is neither complete nor frictionless, and one can find significant discordance between empirical PH behaviour and the risk-neutral-pricing behaviour (see, for example, Bauer et al., 2017; Azimzadeh et al., 2014; and Moenig and Bauer, 2015). Hence, it is necessary to evaluate PHs' welfare assuming a set of preferences that is different than the risk-neutral-pricing strategy. The MV analysis of Markowitz (1952) lays the foundation of modern portfolio theory by formalizing stating the investor's trade-off between the risk and return of a strategy.

In light of this, PHs should take the variance of VAs' payouts into account since a low variance indicates stable cash payouts, which is a virtue for a long-term investment vehicle like VAs. To account for a diversity of PHs' behaviour, we consider three types of PHs, namely a *naive PH*, a *naive PH with exogenous shocks*, and a *sophisticated PH*. The naive PH corresponds to the PH who has inadequate financial literacy or market knowledge and hence dismisses the surrender options embedded in the VA, the naive PH with exogenous shocks corresponds to the PH who has exogenous shocks in her life and surrenders the VA contract sub-optimally at an independent random time, and the sophisticated PH corresponds to the PH with adequate financial literacy and

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[1] A 2% constant fee for the asset under management and 20% HWM fee for the newly created HWM.

[2] The fair fee is also called the *break-even fee*. It is the fee to equate the expected present value of PHs' future cash payouts to their initial premiums.

market knowledge who surrenders the VA contract optimally. For the first two types of PHs, we investigate their welfare by evaluating the mean and variance of the VA payout. For the sophisticated PH, we examine the welfare by tackling the time inconsistency stemming from an MV optimal stopping problem.

By evaluating the welfare of the aforementioned three types of PHs with an MV preference, it will be shown that the HWM fee structure can in general improve PHs' welfare in comparison to the constant and state-dependent fee structures. Specifically, the HWM fee structure is more favourable for PHs who are more risk-averse, more likely to experience exogenous shocks over the course of the policy, and have a shorter contract length. Moreover, the HWM fee structure is more robust in PHs' welfare across a range of risk preferences and is therefore more marketable than the constant and state-dependent fee structures.

The main contributions of this paper are summarized in the following four points:

- We propose a VA with a HWM fee structure and show its merits to stabilize the investment account and reduce the variance of the VA payout.
- The analysis of a VA policy with a HWM fee structure is completed by considering the interest of both parties (PHs and insurers) in the transaction. The vast majority of the VA literature focuses on pricing (and hedging) of a certain VA product. In this paper, we examine a variety of PHs' welfare among a set of VAs with the HWM fee structure (as in the spirit of Steinorth and Mitchell, 2015).
- To study the sophisticated PH's welfare, we formally derive a system of extended Hamilton–Jacobi–Bellman (EHJB) equations corresponding to an MV optimal stopping problem with two state variables over a random horizon. The intricacy stemming from the high dimension and the random horizon is overcome. Note that there are numerous papers addressing stochastic control problems under the MV criterion following the seminal work by Basak and Chabakauri (2010), Björk and Murgoci (2010), and Zhou and Li (2000). However, the literature on MV optimal stopping problems is rather scarce. The present work can help fill this void.
- Note that solving a system of EHJB equations corresponding to an MV optimal control problem can, at times, be reduced to solving a system of ordinary differential equations (ODEs) given the semi-explicit form of the value function. However, for an MV optimal stopping problem with a finite horizon, one can only resort to numerical solutions due to the absence of an explicit solution for the value function. In light of this, we develop an algorithm based on Wang and Forsyth (2011) to solve for the system of EHJB equations. We go a step further than Wang and Forsyth (2011) by establishing the connection between the system of EHJB equations and the proposed algorithm.

This paper is organized as follows. Section 2 formally introduces the VA with a HWM fee structure along with its risk-neutral pricing model. Section 3 presents numerical examples of the fair insurance fees and the corresponding surrender regions for a risk-neutral-pricing PH under the HWM fee structure. For comparative purposes, we also present the corresponding results for the constant and state-dependent fee structures. Sections 4, 5, and 6 evaluate the welfare of a naive PH, a naive PH with exogenous shocks, and a sophisticated PH, respectively, under the HWM fee structure. Once again, a comparative analysis is conducted to investigate PHs' welfare under the constant and state-dependent fee structures. Section 7 concludes. Proofs and some technical results are in the appendices.

# 2 The Risk-Neutral Pricing Model

In this section, we give a detailed description of the VA policy under consideration and formulate its risk-neutral pricing model within a continuous-time stochastic optimal stopping framework.

## 2.1 The Contract

We consider a VA policy with maturity $T > 0$. At inception of the contract, the PH pays an initial premium $P$, which is invested in a predetermined basket of funds, namely the policy fund. The insurer sets up an investment account $F$ to track the performance of the underlying policy fund with $F_0 = P$. The VA policy is assumed to have the following embedded guarantees:

- **guaranteed minimum death benefit (GMDB)**: upon the PH's death at time $t \in (0,T)$, the policy stipulates that the death benefit is the greater of the investment account $F_t$ and the guaranteed amount $G_t$.
- **guaranteed minimum maturity benefit (GMMB)**: at the policy maturity $T$, a payment equal to the greater of the investment account $F_T$ and the guaranteed amount $G_T$ is made to the PH.

In what follows, the guaranteed amount is assumed to roll up continuously at a rate $g \geq 0$ before the contract matures; i.e., $G_t = G_0 e^{gt}$ for $t \in (0,T]$. Note that the roll-up rate $g$ should be no greater than the risk-free rate $r$ to exclude arbitrage opportunities.

Meanwhile, the PH is allowed to surrender the VA contract at any time before maturity. If this surrender right is exercised, there are no guarantees applicable and a penalty $\kappa$ is levied on the balance of the investment account. Thus, upon surrender, the PH is to receive $(1-\kappa_t)F_t$ at $t \in [0,T)$. To disincentivize early-surrender behaviour, the surrender penalty $\kappa_t$ is usually assumed to decrease with time.

## 2.2 Evolution of the Investment Account

Consider a probability space $(\Omega,\mathcal{F},\mathbb{Q})$ with a filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ satisfying the usual conditions, where $\mathbb{Q}$ is an equivalent martingale measure. Under the $\mathbb{Q}$ measure, we assume that the policy fund value process $S$ follows a geometric Brownian motion with dynamics

$$\frac{dS_t}{S_t} = r dt + \sigma dW_t^{\mathbb{Q}},$$

where $r > 0$, $\sigma > 0$ and $\{W^{\mathbb{Q}}\}_{t\geq 0}$ is a Brownian motion under $\mathbb{Q}$. Under the HWM fee structure, the insurance fee has two components, namely a continuously charged constant fee with a rate $c \geq 0$ when the investment account is lower than a certain threshold $\vartheta$ and a HWM fee with a rate $\alpha \geq 0$ charged when the investment account reaches new record highs above the threshold $\vartheta$.[3] Under $\mathbb{Q}$, the dynamics of the investment account under the HWM fee structure is assumed to be

$$dF_t = (r - c\mathbb{1}_{\{F_t\leq\vartheta\}})F_t dt + \sigma F_t dW_t^{\mathbb{Q}} - \alpha \mathbb{1}_{\{F_t\geq\vartheta\}} dM_t, \quad (1)$$

[3] Note that the HWM fee structure can be generalized to the case with two thresholds; for example, $\vartheta_1$ and $\vartheta_2$. Only the constant fee $c$ is effective when the investment account is lower than $\vartheta_1$ and only the HWM fee is effective when the investment account reaches new record highs above $\vartheta_2$.

where

$$M_t = \sup_{0 \le s \le t} F_s$$

denotes the HWM of the investment account at time $t$. For comparative purposes, we also introduce the dynamics of the investment account under two other fee structures for VAs:

- the constant fee structure:

$$dF_t = (r - c)F_t dt + \sigma F_t dW_t^{\mathbb{Q}}, \text{ and}$$

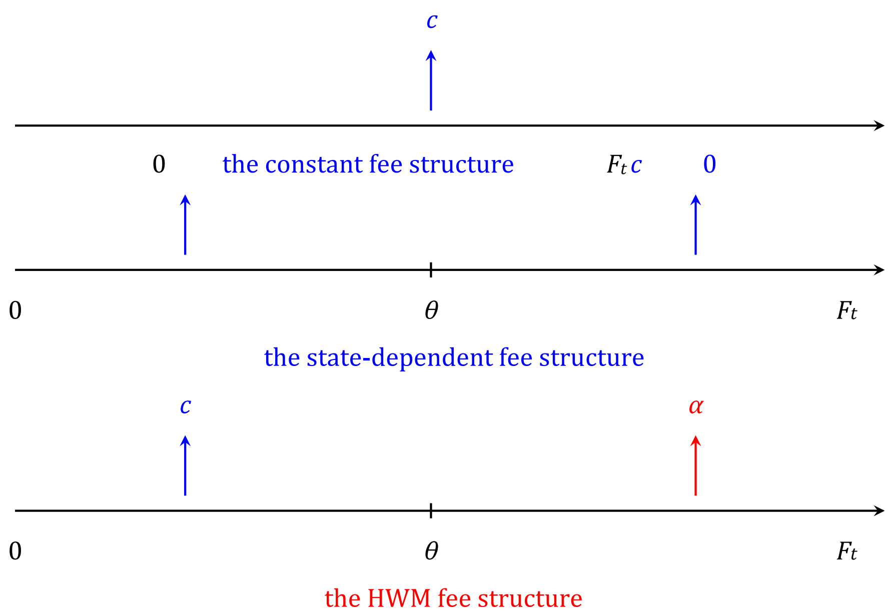
- the state-dependent fee structure (Bernard et al., 2014):

$$dF_t = (r - c\mathbb{1}_{\{F_t \le \theta\}})F_t dt + \sigma F_t dW_t^{\mathbb{Q}}.$$

Note that the same parameter $c$ is used to represent the constant fee in all three fee structures. However, under the risk-neutral pricing model, the parameter $c$ in the HWM fee structure will be different from the ones in the constant or the state-dependent fee structure. We use the same $c$ in all three fee structures for notational convenience.

A graphical illustration of the mechanism of the three fee structures is presented in Figure 1. Note that the HWM fee structure reduces to the state-dependent fee structure if $\alpha = 0$. Also, the constant fee structure can be viewed as a limiting case of the HWM fee structure when $\theta \to \infty$.

**Figure 1: A Graphical Illustration of Fee Mechanism for the Constant, the State-Dependent and the HWM Fee Structures**




## 2.3 The Mortality Model

From the inclusion of the GMDB, the insurer is exposed to mortality risk. Trading only stocks and bonds cannot fully replicate mortality-related insurance claims since mortality-based assets are scarce in the financial market (see Møller, 1998, for details). This gives rise to market incompleteness. To deal with this, it is common in the actuarial literature to make the following two assumptions: financial market risk and mortality risk are independent and the insurer is risk-neutral with respect to mortality risk; see, for instance, Aase and Persson (1994) and Bauer et al. (2008). In this paper, we abide by this convention for the VA pricing. Under the two assumptions, the risk-neutral measure of the combined financial and insurance markets is the product measure of Q and the physical measure of the mortality risk. With a slight abuse of notation, we henceforth denote this product measure by Q.

For convenience, we denote by $\rho_x$ the future lifetime of an x-year-old PH with survival function

$${}_sp_x = \mathbb{P}(\rho_x > s) = e^{-\int_0^s \lambda_{x+u} du}, \text{ for } s \geq 0,$$

where $\lambda_{x+u}$ is the PH's force of mortality at age x + u. In what follows, we omit the subscription x of $\rho_x$ for brevity.

## 2.4 Pricing of the VA Policy

For the pricing of the VA policy, we make use of its resemblance to financial derivatives (e.g., put options) and apply the risk-neutral pricing approach. More specifically, within the risk-neutral pricing framework, both the insurer and the PH are able to replicate every possible VA cash payout. Furthermore, the PH's behaviour is also considered because of the embedded surrender option in the policy. Within the risk-neutral pricing framework, it is opportune for the PH to maximize the EPV of the future cash payouts, a strategy known as the risk-neutral-pricing strategy. The reason is that cash payouts generated by other PH preferences can also be replicated and will result in a lower value than the one corresponding to the risk-neutral-pricing strategy. In light of the above, we formulate the pricing of the VA policy within a continuous-time stochastic optimal stopping framework, and the PH is assumed to surrender the contract optimally to maximize the EPV of the future cash payouts.

Denote the set of all stopping times $\tau$ valued in $[t,T]$ by $T_{[t,T]}$. For the HWM fee structure, the VA's cash payouts are path-dependent and hence the state variable $m$ defined as

$$M_t = \sup_{0 \le s \le t} F_s = m,$$

is further added to preserve the Markov property. Let $\bar{O}$ be the closure of the set $\mathcal{O} = \{(F,m) \in \mathbb{R}_+^2 : 0 < F < m\}$ and consider the value function $V$ of the VA policy at time $t$ over the domain $(t,F,m) \in [0,T] \times \bar{O}$:

$$V(t,F,m) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}^{\mathbb{Q}}_{t,F,m} \left[ \underbrace{{}_{\tau-t}p_{x+t} \psi(\tau, F_\tau) e^{-r(\tau-t)}}_{(a)} + \int_t^\tau \underbrace{{}_{s-t}p_{x+t} \lambda_{x+s} \max(F_s, G_s) e^{-r(s-t)} \, ds}_{(b)} \right], \qquad (2)$$

where

$$\psi(t,F_t) = \begin{cases} (1-\kappa_t)F_t, & 0 \le t < T, \\ \max(G_T, F_T), & t = T, \end{cases}$$

corresponds to the surrender value or the maturity payout of the VA policy. Note that in Eq. (2), (a) contains the surrender and maturity payouts while (b) contains the payout triggered by the PH's death. Let

$$C_t = \{(F,m) \in \bar{O} : V(t,F,m) > \psi(t,F)\}$$

be the continuation region and

$$S_t = \{(F,m) \in \bar{O} : V(t,F,m) = \psi(t,F)\}$$

be the stopping region. We also define

$$\mathcal{L}V := (r - c\mathbb{1}_{\{F<\theta\}})FV_F + \frac{1}{2}\sigma^2 F^2 V_{FF} - (\lambda_{x+t} + r)V,$$

where $V_t$, $V_F$, and $V_{FF}$ are the first-order derivative of $V$ with respect to $t$, $F$, and second-order derivative with respect to $F$, respectively. By the dynamic programming principle, the value function $V$ on $C_t$ satisfies the partial differential equation (PDE)

$$V_t + \mathrm{L}V + \lambda_{x+t} \max(F,G) = 0, \tag{3}$$

with boundary conditions

$$\begin{cases} V(T,F,m) = \max(F,G_T), & \text{for} \quad (F,m) \in [0,m] \times [0,\infty), \\ V_t\big|_{F=0} = (\lambda_{x+t} + r)V\big|_{F=0} - \lambda_{x+t}G, & \text{for} \quad (t,m) \in [0,T) \times [0,\infty), \\ \frac{\partial V}{\partial m}\big|_{F=m} = \alpha \mathbb{1}_{\{F \geq \theta\}} \frac{\partial V}{\partial F}\big|_{F=m}, & \text{for} \quad (t,m) \in [0,T) \times [0,\infty), \\ \lim_{m\to\infty} \frac{V(t,m,m)}{\psi(t,m)} = 1, & \text{for} \quad t \in [0,T). \end{cases} \tag{4}$$

The first boundary condition in Eq. (4) corresponds to the maturity payout of the VA policy. The second boundary condition corresponds to the case that the investment account is ruined; i.e., the degeneration of Eq. (3) when $F = 0$. The third boundary condition is intuitively based on the change of the value of the VA policy when the investment account creates a new HWM. A similar condition can be found in Goetzmann et al. (2003), Lan et al. (2013), and Panageas and Westerfield (2009). The last boundary condition holds as it is optimal for the PH to surrender the contract for large investment account values when the HWM fee $\alpha$ is positive ($\alpha > 0$).[4] A verification theorem for the pricing PDE (3) with boundary conditions (4) is in Theorem 3 in Appendix A.1.

Note that the state variables $F$ and $m$ in the PDE (3) are such that $F \leq m$, which is not desirable to numerically solve the PDE (3). To disentangle the dependence between these two variables, we consider the following change of variable:

$$z = z(F,m) = \frac{F}{m}$$

where $z \in [0,1]$. Evidently, the state variable $z$ represents a drawdown[5] measure of the investment account with respect to its HWM.

[4] Note that the HWM fee structure degenerates to the state-dependent fee structure when the HWM fee $\alpha$ becomes zero. In this case, it is optimal for the PH to hold the VA contract for large investment account values as suggested in MacKay et al. (2017). Therefore, the last boundary condition in Eq. (4) becomes $\lim_{m\to\infty} \frac{V(t,m,m)}{m} = 1$.

[5] Drawdown is a kind of risk metric measuring the magnitude of the decline in portfolio value relative to its historic HWM.

Define a function

$$J(t,z,u) = V(t,F,m)$$

and derive that

$$\begin{cases} V_t = J_t, \\ V_F = \frac{\partial J}{\partial z}\frac{\partial z}{\partial F} + \frac{\partial J}{\partial u}\frac{\partial u}{\partial F} = \frac{1}{m}J_z = \frac{1}{u}J_z, \\ V_{FF} = \frac{\partial V_F}{\partial F} = \frac{1}{u}\left(\frac{\partial J_z}{\partial z}\frac{\partial z}{\partial F} + \frac{\partial J_z}{\partial u}\frac{\partial u}{\partial F}\right) = \frac{1}{u^2}J_{zz} + 0 = \frac{1}{u^2}J_{zz} \\ V_m = \frac{\partial V}{\partial m} = \frac{\partial J}{\partial z}\frac{\partial z}{\partial m} + \frac{\partial J}{\partial u}\frac{\partial u}{\partial m} = J_z\frac{-F}{m^2} + J_u = -\frac{z}{u}J_z + J_u, \end{cases},$$

it is not difficult to see that the PDE (3) with boundary conditions (4) becomes

$$J_t + \mathcal{L}J + \lambda_{x+t}\max(mz,G) = 0, \tag{5}$$

with boundary conditions

$$\begin{cases} J(T,z,m) = \max(zm,G), & \text{for } (z,m) \in [0,1] \times [0,\infty), \\ J_t\big|_{z=0} = (\lambda_{x+t} + r)J\big|_{z=0} - \lambda_{x+t}G, & \text{for } (t,m) \in [0,T) \times [0,\infty), \\ \frac{\partial J}{\partial m}\big|_{z=1} = \frac{\alpha \mathbb{1}_{\{m \geq \theta\}}+1}{m}\frac{\partial J}{\partial z}\big|_{z=1}, & \text{for } (t,m) \in [0,T) \times [0,\infty), \\ \lim_{m\to\infty}\frac{J(t,1,m)}{\psi(t,m)} = 1, & \text{for } t \in [0,T) \end{cases} \tag{6}$$

where $\mathcal{L}J = (r - c\mathbb{1}_{\{mz<\theta\}})zJ_z + \frac{1}{2}\sigma^2 z^2 J_{zz} - (\lambda_{x+t} + r)J$. We propose to numerically solve the above PDE (5) with boundary conditions (6) (rather than Eq. (3) with boundary conditions (4)).

# 3 Fair Fees and Optimal Surrender Regions

In this section, we conduct several numerical experiments of the VA with the HWM fee structure to determine the fair insurance fees and the corresponding optimal surrender regions for a risk-neutral-pricing PH. We carry a comparative analysis of identical VAs, but with the other two fee structures presented in Section 2.2. We later compare their optimal surrender regions to draw implications of a risk management nature for the insurer.

## 3.1 The Fair Fees

For a VA policy, a fair fee is one that makes the EPV of the cash inflows and outflows perfectly balanced. Specifically, we present the following definition of the fair insurance fees for the VA with a HWM fee structure.

**Definition 1.** *For a VA with the HWM fee structure,* $(c,\alpha)$ *is called a pair of fair fees if it satisfies*

$$V\,(0,F_0,F_0;c,\alpha) = F_0.$$

For illustration purposes, we consider the same Makeham mortality model as in MacKay et al. (2017); i.e.,

$$\lambda_x = A + Bc^x, \text{ for } x > 0,$$

with $A = 0.0001$, $B = 0.00035$, and $c = 1.075$. The other parameter inputs are as specified in Table 1. Note that we assess the sensitivity of the fair fees with respect to different contract lengths $T$ and different levels of market volatility $\sigma$. The threshold $\vartheta$ is set at 150 in the following numerical examples, but other values could have as easily been considered. There are various options to choose a surrender penalty function (other than the one stated in Table 1). Contrary to MacKay et al. (2017), we do not look for the minimal surrender penalty that makes surrender behaviour sub-optimal. We adopt the same surrender penalty for all three fee structures so that the fair fees and surrender regions under these fee structures can be fairly compared.

**Table 1: Parameter Inputs**

| Description | Parameter | Value |
|---|---|---|
| Contract length | $T$ | 10, 25 |
| Interest rate | $r$ | 0.03 |
| Roll-up rate | $g$ | 0 |
| Initial premium | $F_0$ | |
| Surrender penalty | $\kappa_t$ | 100<br>0.$05 \times (1 - \frac{t}{T})^3$ |
| The PH's age | $x$ | 60 |
| Volatility | $\sigma$ | 0.15, 0.2, 0.25 |
| Threshold for the constant and HWM fee | $\vartheta$ | 150 |

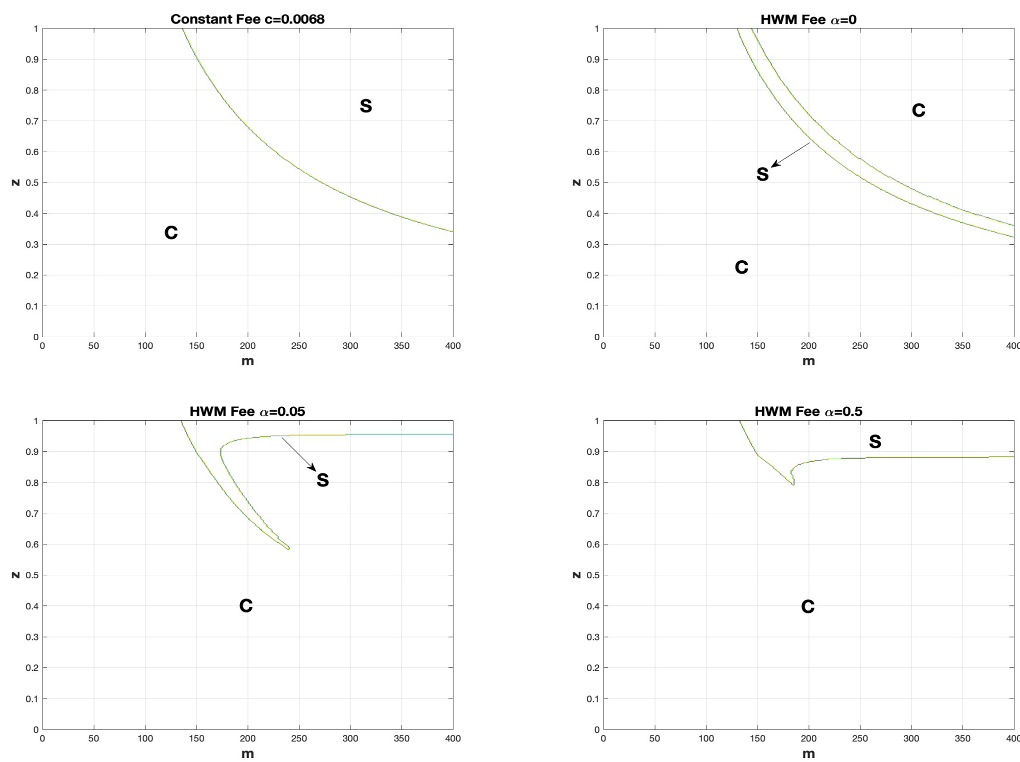
**Table 2: Fair Fee under Different Fee Structures**

| | ***T* = 10** | | | ***T* = 25** | | |
|---|---|---|---|---|---|---|
| **Fee structure** | ***σ* = 0.15** | ***σ* = 0.2** | ***σ* = 0.25** | ***σ* = 0.15** | ***σ* = 0.2** | ***σ* = 0.25** |
| Constant $c$ | 0.0163 | 0.0332 | 0.0550 | 0.0068 | 0.0158 | 0.0278 |
| | (0,0.0170) | (0,0.0338) | (0,0.0555) | (0,0.0087) | (0,0.0182) | (0,0.0307) |
| HWM ($a$; $c$) | (0.05,0.0164) | (0.05,0.0333) | (0.05,0.0550) | (0.05,0.0069) | (0.05,0.0157) | (0.05,0.0277) |
| | (0.2,0.0162) | (0.2,0.0331) | (0.2,0.0547) | (0.2,0.0061) | (0.2,0.0148) | (0.2,0.0263) |
| | (0.5,0.0161) | (0.5,0.0329) | (0.5,0.0543) | (0.5,0.0056) | (0.5,0.0139) | (0.5,0.0247) |

A list of fair fees under different fee structures is summarized in Table 2. In what follows, we refer to the case of a HWM fee $\alpha$ = 0 to the state-dependent fee structure. In Table 2, we consider four levels of HWM fee $\alpha$, namely 0, 0.05, 0.2, and 0.5, and derive the corresponding constant fee $c$. As expected, we first observe that, for a given maturity $T$, the constant fee $c$ decreases as the HWM fee $\alpha$ increases. This is immediate from Definition 1. Now focusing on the time horizon effect, we note that, for a given HWM fee $\alpha$, the fair constant fee $c$ decreases from the case $T$ = 10 to $T$ = 25. This is intuitive as the value of these guarantees tends to decrease with the time horizon. Lastly, for the same HWM fee $\alpha$, higher volatility corresponds to a higher constant fee $c$ (all else being equal). This confirms our intuition that more fees are needed to finance the guarantees in more volatile market conditions.

**Figure 2: Optimal Surrender Regions under Different Fee Structures**

*We present the optimal surrender regions of the risk-neutral-pricing PH under the constant, state-dependent and HWM fee structures at time* t = *12.*



## 3.2 Optimal Surrender Regions

In this subsection, we further analyze the surrender behaviour of the risk-neutral-pricing PH by depicting the corresponding optimal surrender regions. For comparative purposes, we also present the surrender regions for the constant and state-dependent fee structures.

In Figure 2, we present the optimal surrender regions of the risk-neutral-pricing PH at time $t = 12$ under different fee structures for the example of Section 3.1 with contract length $T = 25$ and market volatility $\sigma = 0.15$. In this figure, "**S**" corresponds to the surrender region and "**C**" corresponds to the continuation region. The upper left panel of Figure 2 corresponds to the optimal surrender region of the constant fee structure. Note that the surrender region appears in the upper right corner of the panel, which corresponds to large values of the investment account. This is consistent with the earlier observation that PHs are incentivized to surrender the VA when the investment account becomes large as the value of the guarantees is small in comparison to the EPV of the future insurance fees. The upper right panel of Figure 2 is the optimal surrender region under the state-dependent fee structure ($\alpha = 0$), which corresponds to a banded-shape area below the threshold $\vartheta$ (consistent with MacKay et al., 2017).

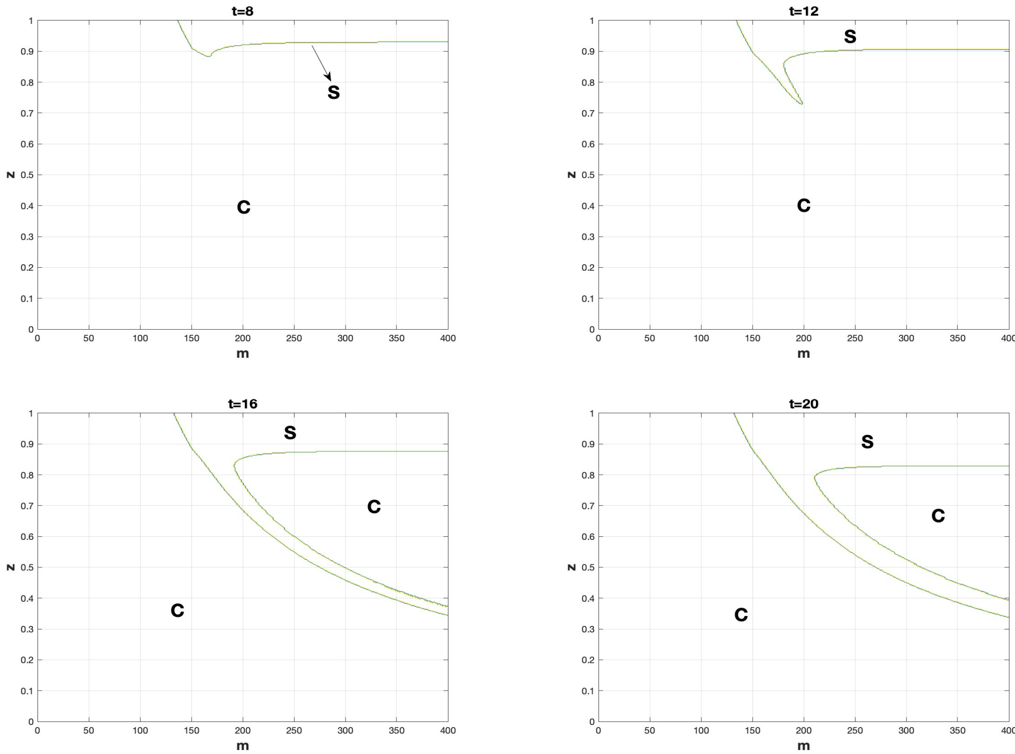
When the investment account is in the banded-shape region, it is optimal for the PH to surrender the contract since the insurance fee is too expensive for the value of the embedded guarantees. Above the threshold $\vartheta$, the PH has no incentive to surrender the contract since no insurance fee is paid. The lower two panels of Figure 2 present the surrender regions of the HWM fee structure with HWM fee $\alpha = 0.05$ (left) and 0.5 (right). For the HWM fee structure, by the inherent design, its surrender region is shown to exhibit features of both the state-dependent fee structure and the HWM fee, namely a banded-shape region below $\vartheta$ and an upper right corner region beyond $\vartheta$. The banded-shape region is attributed to the presence of the state-dependent fee component while the upper right corner region is due to the HWM fee component in the fee structure. For the latter, we note that it may be optimal for the PH to surrender when the account value is large (above the threshold $\vartheta$) due to the possible payment of the HWM fee $\alpha$.

From Figure 2, we also highlight the following features. Firstly, higher HWM fee $\alpha$ (and consequently lower constant fee $c$) discourages the PH to surrender the VA contract when the investment account is lower than the threshold $\vartheta$. This is intuitive as the insurance fee paid is reduced and as such there is less incentive for the PH to surrender the VA policy. This explains the gradual subsiding of the lower part of the banded-shape surrender region as the HWM fee $\alpha$ increases. Secondly, a higher HWM fee incentivizes the PH to surrender the contract when the investment account is higher than the threshold $\vartheta$ since it charges more when the investment account performs well. This explains why increasing the HWM fee $\alpha$ enlarges the surrender regions in the upper right corner of the graphs.

Finally, in Figure 3 we present the optimal surrender regions at times $t = 8$, $t = 12$, $t = 16$, and $t = 20$ for a HWM fee $\alpha = 0.2$ with contract length $T = 25$ and market volatility $\sigma = 0.15$. We observe that the PH is more likely to surrender the contract when the VA policy moves closer to maturity, which is reflected by the enlargement of surrender regions over time. The reason is that the PH is less patient since the embedded guarantees are generally less valuable as we get closer to maturity (all else being equal).

**Figure 3: Optimal Surrender Regions at Different Times**

*Figure 3 presents the optimal stopping regions for a HWM fee α* = 0.2 *at different times.*



# 4 The Naive Policyholder (Type 1)

For the rest of the paper, we examine the impact of the VA with a HWM fee structure on a PH who quantifies her welfare according to an MV criterion. For comparative purposes, the PH's welfare under the constant and state-dependent fee structures is also considered. Specifically, the PH is assumed to evaluate the welfare gained from the VA contract by measuring both the mean and variance of the VA payout. The MV analysis is the cornerstone of modern portfolio theory. Here, a relatively low variance of the VA payouts (relative to its mean) is usually desirable, in particular for risk-averse investors who put an accrued importance on the stability of their investment.

To encompass a variety of PH behaviour, we consider the following three types of PHs: a naive PH (Type 1), a naive PH with exogenous shock (Type 2), and a sophisticated PH (Type 3). More specifically, Type 1 corresponds to a PH who never surrenders the contract, Type 2 corresponds to a PH who surrenders the contract at an independent exponential time (due to exogenous shocks, for example), and Type 3 corresponds to a PH who optimally surrenders the contract. In this section, we focus on evaluating the welfare of the Type 1 PH. The welfare of the Type 2 and Type 3 PHs will be treated separately in Sections 5 and 6, respectively. The rationale for evaluating the welfare of the Type 1 PH is that VAs are complex life insurance products which are known to impede people's ability to measure their values (see Brown et al., 2017). PHs without adequate financial literacy or market knowledge may not understand VAs well enough to optimally exercise their surrender rights. Dismissing the option to surrender the contract (i.e., the Type 1 PH) is one kind of extreme behaviour that is typical for naive PHs (inadequate financial literacy or market knowledge). This behaviour is also a reflection of the empirical observation that PHs exercise VA contracts less often than the risk-neutral-pricing behaviour suggests.

Suppose that the dynamics of the underlying policy fund under the physical probability measure P follows a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

where $W_t$ is a Brownian motion under the measure P. As mentioned above, the Type 1 PH fails to make use of the embedded surrender option. In other words, the contract expires with a payment either at the time of death or at maturity. We assume that the welfare of the Type 1 PH is quantified by the following MV value function $V$ that penalizes for the variance of the VA payout:

$$V(t,F,m) = \mathbb{E}_{t,F,m}\left[e^{-\zeta(\rho\wedge T)}\max(F_{\rho\wedge T}, G_{\rho\wedge T})\right] - \frac{\gamma(F)}{2}\mathrm{Var}_{t,F,m}\left[e^{-\zeta(\rho\wedge T)}\max(F_{\rho\wedge T}, G_{\rho\wedge T})\right], \quad (7)$$

where $\gamma(\cdot)$ corresponds to the state-dependent risk aversion function, $\zeta$ is the subjective discount rate of the PH, and $\rho$ stands for the PH's future lifetime. In what follows, we adopt a similar state-dependent risk aversion function as in Björk et al. (2014); that is,

$$\gamma(F) = \frac{\gamma}{\max(F,G)}, \quad (8)$$

where $\gamma$ is the constant level of risk aversion of the PH. In contrast to Björk et al. (2014), the choice of the denominator is to adapt to the particular context of the VA with embedded guarantees. We note that Eq. (7) can be rewritten as

$$V(t,F,m) = \nu^1(t,F,m) - \frac{\gamma(F)}{2}\left[\nu^2(t,F,m) - \left(\nu^1(t,F,m)\right)^2\right],$$

where $v^1$ and $v^2$ correspond to the first and second moments of the VA payout, respectively; that is,

$$\nu^p(t,F,m) = \mathbb{E}_{t,F,m}\left[e^{-p\zeta(\rho\wedge T)} \max(F_{\rho\wedge T}, G_{\rho\wedge T})^p\right], \text{ for } p = 1,2.$$

For the Type 1 PH, we further have

$$\nu^p(t,F,m) = \mathbb{E}_{t,F,m}\left[{}_{T-t}p_{x+t}e^{-p\zeta(T-t)}\max\left(F_T, G_T\right)^p + \int_t^T {}_{s-t}p_{x+t}\lambda_{x+s}\max(F_s,G_s)^p e^{-p\zeta(s-t)}ds\right],$$

for $p = 1,2$. We propose to solve the PDEs for $v^1$ and $v^2$ to obtain $V$. By the Feynman–Kac formula, for $(t,F,m) \in (0,T) \times O$, $v^p$ ($p = 1,2$) satisfies

$$\frac{\partial \nu^p}{\partial t} + \mathcal{L}\nu^p + \lambda_{x+t}\max\left(F,G\right)^p = 0, \tag{9}$$

for $\mathcal{L}\nu^p = (\mu - c\mathbb{1}_{\{F<\theta\}})F(\nu^p)_F + \frac{1}{2}\sigma^2F^2(\nu^p)_{FF} - (\lambda_{x+t}+\zeta)\nu^p$, with boundary conditions

$$\begin{cases} \nu^p(T,F,m) = \max(F,G_T)^p, & \text{for } (F,m) \in [0,m] \times [0,\infty), \\ \frac{\partial \nu^p}{\partial t}|_{F=0} = (\lambda_{x+t}+\gamma)\nu^p|_{F=0} - \lambda_{x+t}G^p, & \text{for } (t,m) \in [0,T) \times [0,\infty), \\ \frac{\partial \nu^p}{\partial m}\big|_{F=m} = \alpha\mathbb{1}_{\{F\geq\theta\}}\frac{\partial \nu^p}{\partial F}\big|_{F=m}, & \text{for } (t,m) \in [0,T) \times [0,\infty). \end{cases} \tag{10}$$

Henceforth, we conduct all numerical examples with $\mu = 8\%$, $\sigma = 0.15$, and $\zeta = 0.05$. All other parameter inputs are as in Table 1. In what follows, the mean and variance of the VA payout as well as the PH's welfare are all evaluated at the point $(t,F,m) = (0,F_0,F_0)$ and the fair fees are as given by Table 2.

Table 3 summarizes the mean and variance of the VA payout together with the welfare for a naive PH (Type 1) with three different levels of risk aversion. Two time horizons are considered, namely $T = 10$ and $T = 25$.

**Table 3: Welfare of the Naive PH**

*Table 3 summarizes the mean and variance of the VA payout as well as the PH's welfare for three risk aversion levels under different fee structures. Two time horizons are being considered (*$T = 10$ *and* $T = 25$*). The welfare is evaluated by the MV value function V in Eq. (7) at the point* $(t,F,m) = (0,F_0,F_0)$*. The highest mean and welfare and the lowest variance are highlighted in bold.*

| Fee structure | $E_{t,F,m}[\cdot]$ | $Var_{t,F,m}[\cdot]$ | Welfare $\gamma = 0.6$ | Welfare $\gamma = 1.2$ | Welfare $\gamma = 1.8$ |
|---|---|---|---|---|---|
| ***T* = 10** | | | | | |
| Constant | 114.83 | 3479.10 | 104.39 | 93.95 | 83.51 |
| HWM $\alpha = 0$ | **121.21** | 4732.30 | **107.01** | 92.81 | 78.62 |
| HWM $\alpha = 0.05$ | 118.94 | 4110.10 | 106.61 | 94.28 | 81.95 |
| HWM $\alpha = 0.2$ | 112.93 | 2856.60 | 104.36 | **95.79** | 87.22 |
| HWM $\alpha = 0.5$ | 105.40 | **1700.10** | 100.30 | 95.20 | **90.10** |
| *T* = 25 | | | | | |
| Constant | 140.27 | 9595.10 | 111.48 | 82.69 | 53.91 |
| HWM $\alpha = 0$ | **147.79** | 11475.00 | **113.37** | 78.94 | 44.52 |
| HWM $\alpha = 0.05$ | 142.77 | 9829.20 | 113.29 | 83.80 | 54.31 |
| HWM $\alpha = 0.2$ | 127.98 | 6185.00 | 109.43 | 90.87 | 72.32 |
| HWM $\alpha = 0.5$ | 109.75 | **2856.90** | 101.74 | **92.60** | **84.03** |

We consider the HWM fee structure with the HWM fee $\alpha$ ranging from 0 to 0.5, where the case $\alpha = 0$ corresponds to the state-dependent fee structure. For completeness, the results for the constant fee structure are also included.

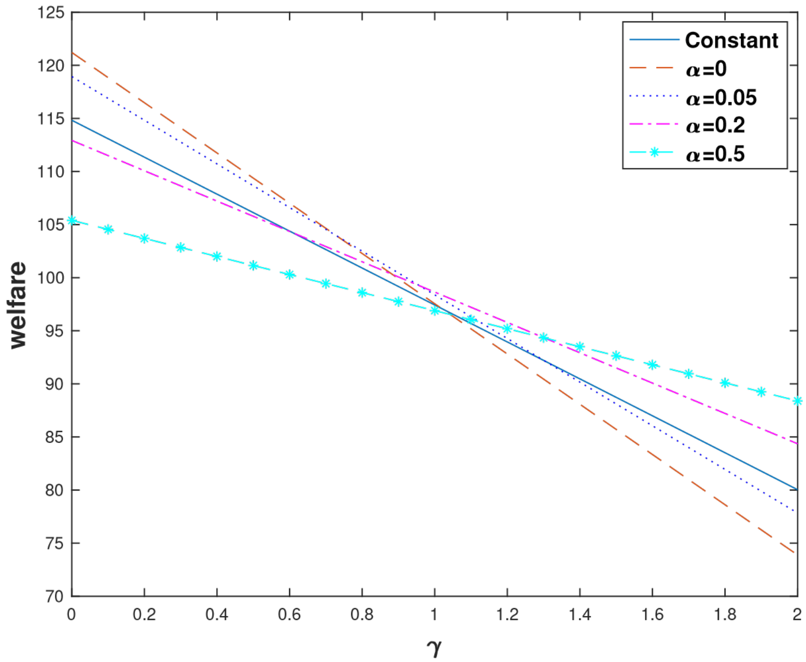
There are a few points worthy of mention here. Firstly, we observe that the reduction in the variance of the VA payout is accompanied by a reduction in its mean when the HWM fee $\alpha$ increases. Indeed, as the HWM fee $\alpha$ increases (and the constant fee $c$ decreases), less (resp. more) insurance fees are charged when the investment account is low (resp. high), which indicates that the investment account is less likely to achieve extreme high or low values.

Secondly, the HWM fee structure can be seen to increase a naive PH's welfare in some cases. More specifically, the HWM fee structure benefits naive PHs who are sufficiently risk-averse. For the low level of risk aversion ($\gamma = 0.6$), we can see from Table 3 that the PH's welfare is the highest under the state-dependent fee structure. However, for a slightly higher level of risk aversion ($\gamma = 1.2$ or 1.8), the PH's welfare is the highest under the HWM fee structure with a HWM fee of $\alpha = 0.2$ and 0.5, respectively.[6] This can be attributed to the fact that PHs with different levels of risk aversion exhibit different attitudes towards the risk as measured by the variance of the VA payout. More specifically, more risk-averse PHs strive to reduce the variance of the VA payout even if this reduction is accompanied by a reduction in the expected payout (as their overall welfare increases).

[6] The increase of the PH's welfare is not always monotone in $\alpha$. One exception is when $T = 10$ and $\gamma = 1.2$, where the PH's welfare peaks at a HWM fee of $\alpha = 0.2$ and it slightly dips for a higher $\alpha$. Nevertheless, this does not contradict the conclusion that the HWM fee structure benefits naive PHs who are more risk-averse.

**Figure 4: Welfare of Naive PHs across Different Risk Aversions**

*Figure 4 depicts the welfare of naive PHs (Type 1) for different risk aversion levels under different fee structures.*




To further examine the impact of the risk aversion level, we depict in Figure 4 the naive PH's welfare for a range of risk aversion levels for a time horizon of $T = 10$. We consider all five fee structures analyzed above. We notice that the Type 1 PH's welfare is less sensitive to the risk aversion level $\gamma$ for a higher HWM fee $\alpha$. This is attributed to the fact that a higher HWM fee $\alpha$ corresponds to a lower VA payout variance and consequently generates a flatter downward trend of PHs' welfare when increasing the risk aversion level $\gamma$. Therefore, we can conclude that the HWM fee structure helps stabilize the welfare of a group of naive PHs across different levels of risk aversion $\gamma$. In other words, the HWM fee structure is more robust for a large group of naive PHs with a variety of risk aversion levels than both the constant and state-dependent fee structures. This feature makes the VA more marketable as it lessens the inequality in welfare among naive PHs with different risk preferences.

## 5 The Naive Policyholder with Exogenous Shocks (Type 2)

In this section, we continue the welfare analysis for a naive PH with exogenous shocks (Type 2). Once again, results for the constant and state-dependent fee structures are also provided for comparative purposes. As mentioned above, a naive PH may completely dismiss the surrender option embedded in the VA contract due to a lack of financial literacy or market knowledge. Nevertheless, the PH may experience certain shocks in their life that could trigger surrendering the policy to access liquidity. These shocks are normally exogenous to the financial market, namely exogenous to the performance of the investment account. Therefore, to model the Type 2 PH's behaviour, we assume that the arrival time $\tau$ of a certain exogenous shock follows an exponential distribution with mean $\frac{1}{\eta}$, independently of the PH's future lifetime $\rho$ as well as the investment account $\{F_t\}_{t\in[0,T]}$. We assume that the PH surrenders the contract upon the arrival of the shock.

Note that the VA contract will either pay a surrender, death, or maturity benefit. Therefore, as in Section 4, the Type 2 PH's welfare is quantified by the following MV value function $V$ defined as:

$$V(t,F,m)=\mathbb{E}_{t,F,m}\left[e^{-\zeta(\tau\wedge\rho\wedge T)}H(F_{\tau\wedge\rho\wedge T})\right]-\frac{\gamma(F)}{2}\mathrm{Var}_{t,F,m}\left[e^{-\zeta(\tau\wedge\rho\wedge T)}H(F_{\tau\wedge\rho\wedge T})\right], \tag{11}$$

where

$$H(F_{\tau\wedge\rho\wedge T})=\begin{cases}(1-\kappa_\tau)F_\tau, & \tau<\rho\wedge T,\\ \max(G_\rho,F_\rho), & \rho<\tau\wedge T,\\ \max(G_T,F_T), & T<\tau\wedge\rho,\end{cases} \tag{12}$$

corresponds to the contract payouts. As before, we make use of the first and second moments of the VA payout $\nu^1$ and $\nu^2$ to facilitate the quantification of the Type 2 PH's welfare. The expressions of the first and second moments $\nu^1$ and $\nu^2$ of the VA payout are presented in the following proposition.

**Proposition 1.** *For the Type 2 PH who surrenders the contract following an independent exponential distribution at rate $\eta>0$, the first and the second moments of the VA payout are respectively given by*

$$\nu^p(t,F,m)=\mathbb{E}_{t,F,m}\left[e^{-\int_t^T\delta_s^p ds}\max(F_T,G_T)^p+\int_t^T e^{-\int_t^s\delta_u^p du}\left(\eta\left[(1-\kappa_s)F_s\right]^p+\lambda_{x+s}\max(F_s,G_s)^p\right)ds\right]$$

*where $\delta_t^p = p\zeta+\lambda_{x+t}+\eta$ for $p=1,2$.*

The proof of Proposition 1 is in Appendix A.2. By the Feynman–Kac formula, we can derive the corresponding PDEs for $\nu^1$ and $\nu^2$ in the following. For $(t,F,m)\in(0,T)\times O$, $\nu^p$ ($p=1,2$) satisfies

$$\frac{\partial\nu^p}{\partial t}+\mathcal{L}\nu^p+\eta[(1-\kappa_t)F]^p+\lambda_{x+t}\max(F,G)^p=0, \tag{13}$$

with boundary conditions

$$
\begin{cases}
\nu^p(T,F,m) = \max(F,G_T)^p, & \text{for } (F,m) \in [0,m] \times [0,\infty), \\
\frac{\partial \nu^p}{\partial t}|_{F=0} = \delta^p \nu^p|_{F=0} - \lambda_{x+t} G^p, & \text{for } (t,m) \in [0,T) \times [0,\infty), \\
\frac{\partial \nu^p}{\partial m}\Big|_{F=m} = \alpha \mathbb{1}_{\{F \geq \theta\}} \frac{\partial \nu^p}{\partial F}\Big|_{F=m}, & \text{for } (t,m) \in [0,T) \times [0,\infty),
\end{cases}
\tag{14}
$$

where

$$\mathcal{L}\nu^p = (\mu - c\mathbb{1}_{\{F<\theta\}})F(\nu^p)_F + \tfrac{1}{2}\sigma^2 F^2 (\nu^p)_{FF} - \delta^p \nu^p,$$ and $\delta p = \delta tp$.

Tables 4 and 5 summarize the mean and variance of the VA payout and the corresponding welfare under different risk aversion levels $\gamma$ for a VA with time horizon $T = 10$ and $T = 25$, respectively. Results are provided for the same fee structures as the ones considered in Section 4. The surrender time of the PH follows an independent exponential distribution with surrender intensities $\eta = 0.1$, 1, or 10. Note that the hypothetical case $\eta = 0$ reduces to the Type 1 PH of Section 4. We sum up observations from Tables 4 and 5 in the following.

Firstly, as expected, we observe that the mean and the variance of the VA payout generally decrease with respect to the surrender intensity $\eta$. This is an immediate consequence of the fact that the VA payout is generally made sooner, more likely triggered by a surrender whose payout is reduced by a hefty surrender penalty.

Secondly, for a small surrender intensity $\eta$ (e.g., $\eta = 0.1$ or 1), as in Section 4, the mean and variance of the VA payout are reduced when the HWM fee $\alpha$ increases. However, for a large surrender intensity $\eta$ (e.g., $\eta = 10$), this reduction of the mean when the HWM fee $\alpha$ increases no longer holds. This is because the PH with a large surrender intensity is more likely to surrender the contract, and if so at an early time, making it more difficult for the HWM fee $\alpha$ to become effective.

Thirdly, the HWM fee structure is beneficial for PHs with a significant probability to experience some exogenous shocks during the lifetime of the contract. For a large surrender intensity ($\eta = 10$), the *highest welfare* appears at a HWM fee $\alpha = 0.5$ for all three levels of risk aversion $\gamma$ and both time horizons $T = 10$ and $T = 25$. Lastly, Type 2 PHs' welfare under surrender intensities $\eta = 0.1$ and $\eta = 1$ is comparable to that of Type 1 PHs in the sense that the HWM fee is favourable to Type 2 PHs who are more risk-averse ($\gamma = 1.2$, or 1.8) while the state-dependent fee structure is favourable to Type 2 PHs who are less risk-averse ($\gamma = 0.6$). This can be seen from the highest welfare and the trend of PHs' welfare when increasing the HWM fee $\alpha$ under the two surrender intensities $\eta = 0.1$ or $\eta = 1$.

As in Section 4, we intend to examine a group of Type 2 PHs' welfare across a wide spectrum of risk aversion levels. We refer the reader to Figure 5, which depicts the Type 2 PH welfare as a function of the risk aversion level for different fee structures when the VA has a maturity of $T = 10$. In Figure 5, the surrender intensity was chosen to be $\eta = 0.1$.

**Table 4: Welfare of the Naive PH with Exogenous Shock**

*Table 4 summarizes the mean and variance of the VA payout and the welfare for a Type 2 PH under different fee structures for the time horizon* T *= 10. The welfare is evaluated by the MV value function V in Eq.* (11) *at the point* $(t,F,m) = (0,F_0,F_0)$*. The highest mean and welfare and the lowest variance are highlighted in bold.*

| | $T = 10$ | | | | | |
|---|---|---|---|---|---|---|
| | Fee structure | $\mathbb{E}_{t,F,m}[\cdot]$ | $\mathrm{Var}_{t,F,m}[\cdot]$ | Welfare $\gamma = 0.6$ | Welfare $\gamma = 1.2$ | Welfare $\gamma = 1.8$ |
| | Constant | 108.1601 | 2260.80 | 101.3777 | 94.5953 | 87.8129 |
| $\eta = 0.1$ | $\alpha = 0$ | **111.4847** | 2944.50 | **102.6512** | 93.8177 | 84.9842 |
| | $\alpha = 0.05$ | 110.2841 | 2602.90 | 102.4754 | 94.6667 | 86.8580 |
| | $\alpha = 0.2$ | 106.9499 | 1902.00 | 101.2439 | **95.5379** | 89.8319 |
| | $\alpha = 0.5$ | 102.7070 | **1238.20** | 98.9924 | 95.2778 | **91.5632** |
| | Constant | 97.7186 | 380.7885 | 96.5762 | 95.4339 | 94.2915 |
| $\eta = 1$ | $\alpha = 0$ | **97.7757** | 398.0850 | 96.5814 | 95.3872 | 94.1929 |
| | $\alpha = 0.05$ | 97.7610 | 387.0364 | **96.5999** | 95.4388 | 94.2777 |
| | $\alpha = 0.2$ | 97.6067 | 360.1458 | 96.5256 | **95.4451** | 94.3647 |
| | $\alpha = 0.5$ | 97.3858 | **328.2719** | 96.4010 | 95.4162 | **94.4314** |
| | Constant | 95.2953 | 59.1446 | 95.1179 | 94.9404 | 94.7630 |
| $\eta = 10$ | $\alpha = 0$ | 95.2889 | 59.1227 | 95.1115 | 95.9342 | 94.7568 |
| | $\alpha = 0.05$ | 95.2944 | 59.1142 | 95.1171 | 94.9397 | 94.7624 |
| | $\alpha = 0.2$ | **95.2952** | 59.0206 | 95.1181 | 94.9411 | 94.7640 |
| | $\alpha = 0.5$ | 95.2948 | **58.8727** | **95.1182** | **94.9416** | **94.7649** |

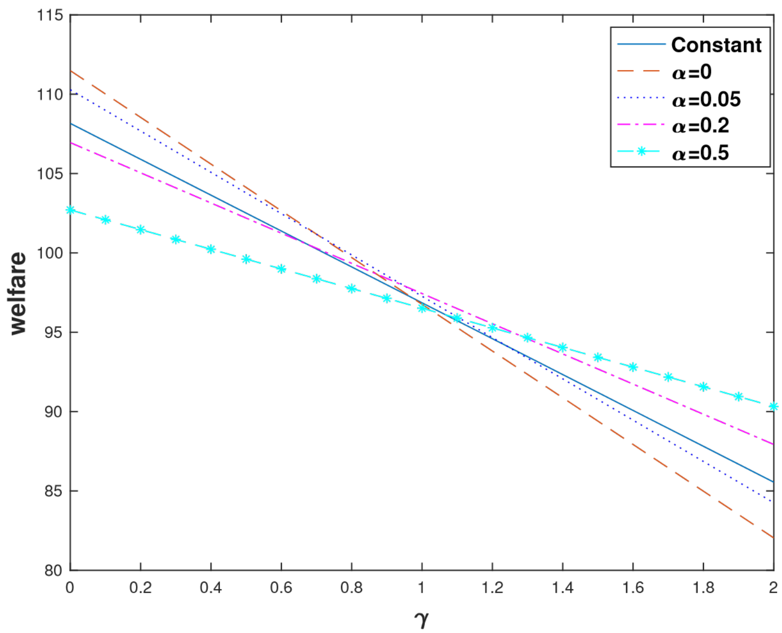
**Table 5: Welfare of the Naive PH with Exogenous Shock**

*Table 5 summarizes the mean and variance of the VA payout and the welfare for a Type 2 PH under different fee structures for the time horizon* T *= 25. The welfare is evaluated by the MV value function V in Eq.* (11) *at the point* $(t,F,m) = (0,F_0,F_0)$*. The highest mean and welfare and the lowest variance are highlighted in bold.*

| | Fee structure | $\mathbb{E}_{t,F,m}[\cdot]$ | $\mathrm{Var}_{t,F,m}[\cdot]$ | Welfare $\gamma = 0.6$ | Welfare $\gamma = 1.2$ | Welfare $\gamma = 1.8$ |
|---|---|---|---|---|---|---|
| $T = 25$ | | | | | | |
| | Constant | 116.6123 | 4099.10 | 104.3150 | 92.0177 | 79.7204 |
| $\eta = 0.1$ | $\alpha = 0$ | **118.7506** | 4787.90 | 104.3869 | 90.0232 | 75.6595 |
| | $\alpha = 0.05$ | 117.1704 | 4131.60 | **104.7756** | 92.3808 | 79.9860 |
| | $\alpha = 0.2$ | 111.8027 | 2737.40 | 103.5905 | 95.3783 | 87.1661 |
| | $\alpha = 0.5$ | 105.0219 | **1513.40** | 100.4817 | **95.9415** | **91.4013** |
| | Constant | **97.9733** | 394.2303 | **96.7906** | 95.6079 | 94.4252 |
| $\eta = 1$ | $\alpha = 0$ | 97.8606 | 399.7407 | 96.6614 | 95.4622 | 94.2629 |
| | $\alpha = 0.05$ | 97.9512 | 390.0008 | 96.7812 | 95.6112 | 94.4412 |
| | $\alpha = 0.2$ | 97.8376 | 362.6018 | 96.7498 | **95.6620** | 94.5742 |
| | $\alpha = 0.5$ | 97.6322 | **329.5321** | 96.6436 | 95.6550 | **94.6664** |
| | Constant | 95.2986 | 59.6349 | 95.1197 | 94.9408 | 94.7619 |
| $\eta = 10$ | $\alpha = 0$ | 95.2809 | 59.5007 | 95.1024 | 94.9239 | 94.7454 |
| | $\alpha = 0.05$ | 95.2981 | 59.6034 | 95.1193 | 94.9405 | 94.7617 |
| | $\alpha = 0.2$ | 95.3049 | 59.5314 | 95.1263 | 94.9477 | 94.7691 |
| | $\alpha = 0.5$ | **95.3083** | **59.4198** | **95.1300** | **94.9518** | **94.7735** |

**Figure 5: Welfare of Naive PHs with Exogenous Shocks across Different Risk Aversions**

*Figure 5 depicts the welfare of naive PHs with exogenous shocks (Type 2) for different risk aversion levels under different fee structures.*




Consistent to the Type 1 PH case, Figure 5 indicates that the HWM fee structure is more robust and consequently more marketable for a group of Type 2 PHs who are heterogeneous in risk preferences in comparison to the other two fee structures. Similarly as in Figure 4, a higher HWM fee $\alpha$ corresponds to a flatter line of welfare, which reflects that Type 2 PHs' welfare is less sensitive to a change in the risk aversion level $\gamma$.

## 6 The Sophisticated Policyholder (Type 3)

In this section, we examine the welfare of a sophisticated PH (Type 3) for the VA with a HWM fee structure. Once again, the corresponding results for the constant and state-dependent fee structures are also presented.

Type 3 corresponds to PHs who surrender the contract optimally according to their preferences, and hence models the behaviour of PHs with adequate financial literacy and market knowledge. We formulate the PH's welfare into a stochastic MV optimal stopping problem under a state-dependent risk aversion (as in Björk et al., 2014). One of the major intricacies of the stochastic MV optimization problem is the well-known concept of time inconsistency (see, for example, Basak and Chabakauri, 2010; Björk and Murgoci, 2014; and Björk et al., 2017). To deal with this, we adopt the game theoretic approach proposed in Björk and Murgoci (2010) and derive an equilibrium stopping strategy for the PH by solving a system of EHJB equations. We point out that the present MV problem is different from the existing ones in the literature in the following aspects.

Firstly, we deal with an MV optimal stopping problem which differs from MV optimal control problems. To apply the game theoretic framework in Björk and Murgoci (2010), we transform the optimal stopping problem into an optimal control problem with a binary control.

Secondly, by the inherent design of the VA policy with the HWM fee structure, we solve an MV optimal stopping problem with two state variables over a random horizon. The level of complexity of the high-dimensional problem is further accrued by the state-dependent risk aversion which will lead to the introduction of another state variable. Also, the consideration of a random horizon further complicates the derivation of the system of EHJB equations.

With the absence of an explicit solution to the value function, we resort to numerically solving the system of EHJB equations corresponding to the MV optimal stopping problem. We develop an algorithm based on Wang and Forsyth (2011) tailored to solve MV optimal control problems. This algorithm significantly reduces the complexity of directly solving the system of EHJB equations numerically by circumventing some complications arising from the high-dimensional nature of the problem and its iterative numerical procedure. Also, we further establish the connection between the algorithm and the system of EHJB equations by proving that the algorithm is indeed a numerical approximation to the system of EHJB equations.

### 6.1 Model Formulation

Note that the sophisticated PH (Type 3) optimally decides the surrender time to maximize the MV objective with a state-dependent risk aversion. Hence, the value function that quantifies the PH's welfare is

$$V(t,F,m) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \left\{ \mathbb{E}_{t,F,m}\left[ H(F_{\tau\wedge\rho}) \middle| \rho > t \right] - \frac{\gamma(F)}{2} \mathrm{Var}_{t,F,m}\left[ H(F_{\tau\wedge\rho}) \middle| \rho > t \right] \right\} \tag{15}$$

for $(t,F,m) \in [0,T]\times \bar{O}$, where $H$ corresponds to the VA payout defined in Eq. (12), $\mathcal{T}_{[t,T]}$ denotes the set of all stopping times $\tau$ valued in $[t,T]$, and the risk aversion function $\gamma(F)$ is as given in Eq. (8). Note that a stochastic optimal stopping problem can be viewed as an optimal control problem with binary controls (see, for example, Ebert et al., 2018, and Tan et al., 2018). Therefore, we can regard our optimal stopping problem as an optimal control problem by introducing the following stopping rule.

**Definition 2.** *A stopping rule is a measurable function* $u : [0,T] \times \bar{O} \to \{0,1\}$ *where* 0 *corresponds to continuation and* 1 *corresponds to stopping.*

Specifically, in our setup, the continuation and stopping regions at time $t \in [0,T)$ are defined as

$$\begin{cases} \mathcal{C}_t = \{(F,m) \in \bar{\mathcal{O}} : u(t,F,m) = 0\} \\ \mathcal{S}_t = \{(F,m) \in \bar{\mathcal{O}} : u(t,F,m) = 1\} \end{cases},$$

with the corresponding stopping time $\tau^u$ defined as

$$\tau^u = \inf\{s \geq t : u(s,F_s,M_s) = 1\}.$$

In light of the above, Eq. (15) is rewritten as

$$V(t,F,m) = \sup_{\boldsymbol{u} \in \boldsymbol{\mathcal{A}}} \left\{ \mathbb{E}_{t,F,m}\left[ H\left(F_{\tau^{\boldsymbol{u}} \wedge \rho}\right) \middle| \rho > t \right] - \frac{\gamma(F)}{2} \mathrm{Var}_{t,F,m}\left[ H\left(F_{\tau^{\boldsymbol{u}} \wedge \rho}\right) \middle| \rho > t \right] \right\} := \sup_{\boldsymbol{u} \in \boldsymbol{\mathcal{A}}} \mathcal{J}\left(t,F,m;\tau^{\boldsymbol{u}}\right), \tag{16}$$

where $A$ is the admissible set of stopping rules. As discussed earlier, the nature of the stochastic MV optimization problem gives rise to the time inconsistency of the optimal strategy. As is common in the literature (see, for example, Björk and Murgoci, 2010), we propose to overcome the time-inconsistency issue by looking for an equilibrium strategy within the game theoretic framework. Following the definition of an equilibrium strategy over a random horizon in Landriault et al. (2018), we introduce the definition of an equilibrium stopping rule over a random horizon for the problem at interest here.

**Definition 3.** *For a fixed point* $(t,F,m) \in [0,T) \times \bar{O}$*, a small* $\varepsilon > 0$*, and an admissible stopping rule* $\hat{u} = \{\hat{u}_s\}_{s\in[t,T)}$ *conditional on* $\rho > t$*, define a stopping rule* $u^\varepsilon$ *by*

$$u^{\varepsilon}(s,y,z) = \begin{cases} u \in \{0,1\}, & \textit{for} \;\; t \leq s < (t+\varepsilon) \wedge \rho, \\ \hat{u}(s,y,z), & \textit{for} \;\; (t+\varepsilon) \wedge \rho \leq s < T \wedge \rho, \end{cases}$$

*in which* $(y,z) \in [0,m] \times [0,\infty)$*. If*

$$\liminf_{\varepsilon \downarrow 0} \frac{\mathcal{J}(t,F,m;\tau^{\hat{\boldsymbol{u}}}) - \mathcal{J}(t,F,m;\tau^{\boldsymbol{u}^{\varepsilon}})}{\varepsilon} \geq 0$$

*for all* $(t,F,m) \in [0,T) \times \bar{O}$*, then* $\hat{u}$ *is called an equilibrium feedback stopping rule and the corresponding equilibrium value function V is given by* $V(t,F,m) = J(t,F,m;\tau^{\hat{u}})$.

In what follows, we present a verification theorem which justifies that a solution of a system of EHJB equations is a solution of the optimization problem (16). The proof of the theorem is deferred to Appendix A.3.

**Theorem 1.** *Suppose that there exist functions* $V,g \in C^{1,2,1}([0,T] \times \bar{O})$*,* $f \in C^{1,2,1,2}([0,T] \times \bar{O} \times [0,m])$*, and a stopping rule* $\hat{u}$ *satisfying the following conditions:*

- *For any* $(t,F,m) \in [0,T] \times \bar{O}$*, V solves*

$$\min\Big\{ -V_t - \left(\mu - c\mathbb{1}_{\{F\le\theta\}}\right) F\left(V_F - f_y - \frac{\gamma_F}{2}g^2\right) - \frac{1}{2}\sigma^2F^2\Big(V_{FF} - f_{yy} - 2f_{Fy} - \frac{\gamma_{FF}}{2}g^2 - \gamma(F)g_F^2 - 2\gamma_F g g_F\Big) + \frac{\lambda_{x+t}\gamma(F)}{2}\left(g - \max(F,G)\right)^2 + \lambda_{x+t}V - \lambda_{x+t}\max(F,G), V - \psi\Big\} = 0, \tag{17}$$

*with boundary conditions*

$$\begin{cases} V(T,F,m) = \max(F,G), & \textit{for}\ \ (F,m)\in\bar{\mathcal{O}},\\ V_t\big|_{F=0} = \frac{\lambda_{x+t}\gamma(0)}{2}(g\big|_{F=0} - G)^2 + \lambda_{x+t}V\big|_{F=0} - \lambda_{x+t}G, & \textit{for}\ \ (t,m)\in[0,T)\times[0,\infty),\\ V_m\big|_{F=m} = \alpha\mathbb{1}_{\{F\ge\theta\}}\left(V_F\big|_{F=m} - f_y\big|_{F=m} - \frac{\gamma_F}{2}g^2\big|_{F=m}\right), & \textit{for}\ \ (t,m)\in[0,T)\times[0,\infty),\\ \lim_{m\to\infty}\frac{V(t,m,m)}{\psi(t,m)} = 1, & \textit{for}\ \ t\in[0,T). \end{cases} \tag{18}$$

- *For any* $(t,F,m) \in [0,T) \times \bar{O}$,

$$\hat{u}(t,F,m) = \begin{cases} 1, & \textit{if}\ \ V(t,F,m) = \psi(t,F),\\ 0, & \textit{otherwise}. \end{cases} \tag{19}$$

- *For fixed* $y \in [0,m]$ *and for any* $(t,F,m) \in [0,T] \times \bar{O}$ *s.t.* $\hat{u} = 0$, *f solves*

$$f_t + \left(\mu - c\mathbb{1}_{\{F\le\theta\}}\right)Ff_F + \frac{1}{2}\sigma^2F^2f_{FF} - \lambda_{x+t}f + \lambda_{x+t}\left[\max(F,G) - \frac{\gamma(y)}{2}\max(F,G)^2\right] = 0, \tag{20}$$

*with boundary conditions*

$$\begin{cases} f(T,F,m,y) = \max(F,G) - \frac{\gamma(y)}{2}\max(F,G)^2, & \textit{for}\ \ (F,m,y)\in\bar{\mathcal{O}}\times[0,m],\\ f_t\big|_{F=0} = \lambda_{x+t}f\big|_{F=0} - \lambda_{x+t}\left(G - \frac{\gamma(y)}{2}G^2\right), & \textit{for}\ \ (t,m,y)\in[0,T)\times[0,\infty)\times[0,m],\\ f_m\big|_{F=m} = \alpha\mathbb{1}_{\{F\ge\theta\}}f_F\big|_{F=m}, & \textit{for}\ \ (t,m,y)\in[0,T)\times[0,\infty)\times[0,m],\\ \lim_{m\to\infty}\frac{f(t,m,m,y)}{\psi(t,m)-\frac{\gamma(y)}{2}\psi^2(t,m)} = 1, & \textit{for}\ \ t\in[0,T). \end{cases} \tag{21}$$

*And for* $\hat{u} = 1,\ f(t,F,m,y) = \psi(t,F) - \frac{\gamma(y)}{2}\psi^2(t,F)$.

- *For any* $(t,F,m) \in [0,T] \times \bar{O}$ *s.t.* $\hat{u} = 0$, *g solves*

$$g_t + \left(\mu - c\mathbb{1}_{\{F\le\theta\}}\right)Fg_F + \frac{1}{2}\sigma^2F^2g_{FF} - \lambda_{x+t}g + \lambda_{x+t}\max(F,G) = 0, \tag{22}$$

*with boundary conditions*

$$\begin{cases} g(T,F,m) = \max(F,G), & \textit{for}\ \ (F,m)\in\bar{\mathcal{O}},\\ g_t\big|_{F=0} = \lambda_{x+t}g\big|_{F=0} - \lambda_{x+t}G, & \textit{for}\ \ (t,m)\in[0,T)\times[0,\infty),\\ g_m\big|_{F=m} = \alpha\mathbb{1}_{\{F\ge\theta\}}g_F\big|_{F=m}, & \textit{for}\ \ (t,m)\in[0,T)\times[0,\infty),\\ \lim_{m\to\infty}\frac{g(t,m,m)}{\psi(t,m)} = 1, & \textit{for}\ \ t\in[0,T). \end{cases} \tag{23}$$

*And for* $\hat{u} = 1$*,* $g(t,F,m) = \psi(t,F)$*.*

*Then* $\hat{u}$ *is an equilibrium stopping rule such that*

$$\begin{cases} V(t,F,m) = \mathcal{J}(t,F,m;\tau^{\hat{u}}) \\ g(t,F,m) = \mathbb{E}_{t,F,m}\left[H\left(F_{\tau^{\hat{u}}\wedge\rho}\right)|\rho > t\right] \\ f(t,F,m,y) = \mathbb{E}_{t,F,m}\left[H\left(F_{\tau^{\hat{u}}\wedge\rho}\right)|\rho > t\right] - \frac{\gamma(y)}{2}\mathbb{E}_{t,F,m}\left[H\left(F_{\tau^{\hat{u}}\wedge\rho}\right)^2|\rho > t\right]. \end{cases} \tag{24}$$

This system of EHJB equations corresponds to a high-dimensional free boundary problem with three unknown functions $V$, $f$, and $g$, where $\hat{u} = 0$ corresponds to the continuation region and $\hat{u} = 1$ to the stopping region.

## 6.2 The Algorithm

Given the complex nature of the system of EHJB equations (17)–(23), we propose to numerically solve it via the development of an algorithm. Inspired by the work of Wang and Forsyth (2011) on the development of an algorithm for MV optimal control problems, the proposed algorithm will circumvent the complexity arising from directly solving the system of EHJB equations (17)–(23) corresponding to our MV optimal stopping problem. In addition, we go a step further than Wang and Forsyth (2011) by establishing a connection between the system of EHJB equations (17)–(23) and the algorithm.

Directly solving the system of EHJB equations (17)–(23) numerically using a finite difference method roughly entails the following steps:

1. Partition $[0,T]$ into $N$ equal intervals s.t. $\Delta t = \frac{T}{N}$ and $t_n = T - n\Delta t$ for $n = 0,1,...,N$.
2. At time $t_n$, solve $g$ numerically using Eq. (22) with boundary conditions (23) and solve $f$ numerically for each fixed $y \in [0,m]$ using Eq. (20) with boundary conditions (21), by assuming that $\hat{u} = 0$ (no surrender) over the domains $[0,m) \times [0,\infty)$.
3. Substitute the resulting $f$ and $g$ in Eq. (17) and solve $V$ numerically using Eq. (17) with boundary conditions (18) to obtain an equilibrium stopping rule $\hat{u}$ (or equivalently an equilibrium stopping region).
4. Update $f$ and $g$ so that their values correspond to the equilibrium stopping rule $\hat{u}$. Specifically, if $\hat{u}(t_n,F,m) = 0$, keep $f(t_n,F,m,y)$ and $g(t_n,F,m)$ unchanged; if $\hat{u}(t_n,F,m) = 1$, let $f(t_n,F,m,y) = \psi(t_n,F) - \frac{\gamma(F)}{2}\psi^2(t.F)$ and $g(t_n,F,m) = \psi(t_n,F)$.
5. Substitute the updated $f$ and $g$ in Eq. (17) to update $V$.
6. Repeat Steps 4 and 5 until $V$ converges.
7. Move to time $t_{n+1}$ and repeat Steps 2–6.

The challenges of solving the system of EHJB equations (17)–(23) numerically using the above steps are three-fold. Firstly, solving $f$ is time-consuming as it entails solving an unknown function with two state variables for each fixed $y \in [0,m]$. Secondly, the complex form of Eq. (17) makes the computational process highly non-trivial. Thirdly, at each time point $t_n$, a time-consuming iterative procedure is required to solve for $V$ and identify an equilibrium stopping rule $\hat{u}$.

In light of the above, we develop an algorithm based on Wang and Forsyth (2011) to circumvent the complexity of numerically solving the system of EHJB equations (17)–(23) directly. As in Sections 4 and 5, the mean and variance of the VA payout function $H$ can be represented by its first and second moments. Therefore, the value function $V$ in Eq. (16) can be rewritten as

$$V(t,F,m) = \sup_{\boldsymbol{u}\in\mathcal{A}} \left\{ \nu^1(t,F,m;\tau^{\boldsymbol{u}}) - \frac{\gamma(F)}{2}\left[\nu^2(t,F,m;\tau^{\boldsymbol{u}}) - \left(\nu^1(t,F,m;\tau^{\boldsymbol{u}})\right)^2\right]\right\}, \qquad (25)$$

where

$$\nu^p(t,F,m;\tau^{\boldsymbol{u}}) = \mathbb{E}_{t,F,m}\left[H^p\left(F_{\tau^{\boldsymbol{u}}\wedge\rho}\right)\Big|\rho > t\right], \quad \text{for} \quad p = 1,2, \qquad (26)$$

denote the first and second moments of the VA payout function $H$ under a certain stopping rule $u \in A$, respectively. Furthermore, from Eq. (24), we observe that the value functions $V$ , $f$, and $g$ are all linear combinations of $v^1$ and $v^2$ under an equilibrium stopping rule. We also observe that over the domain $[0,T] \times O^-$ s.t. $\hat{u} = 0$, $v^p$ ($p = 1,2$) satisfy Eq. (9) with boundary conditions (10) with $\zeta = 0$. In view of this, instead of numerically solving the system of EHJB equations (17)–(23) in a direct manner, one can get $v^1$ and $v^2$ by solving their corresponding PDEs to obtain an equilibrium stopping rule. We break this procedure into the following steps:

**Step 1:** At time $t_n$, solve $v^p$ ($p = 1,2$) numerically using Eq. (9) with boundary conditions Eq. (10) with $\zeta = 0$, by assuming that $u = 0$ (no surrender) over the domain $O^-$.

**Step 2:** Obtain an equilibrium stopping rule $\hat{u}$ by substituting the corresponding $v^1$ and $v^2$ in Eq. (25).

**Step 3** Update $v^1$ and $v^2$ s.t. if $\hat{u}(t_n,F,m) = 0$, keep $v^1(t_n,F,m)$ and $v^2(t_n,F,m)$ unchanged; if $\hat{u}(t_n,F,m) = 1$, let $v^1(t_n,F,m) = \psi(t_n,F)$ and $v^2(t_n,F,m) = \psi^2(t_n,F)$.

**Step 4** Move to time $t_{n+1}$ and repeat **Steps 1–3**.

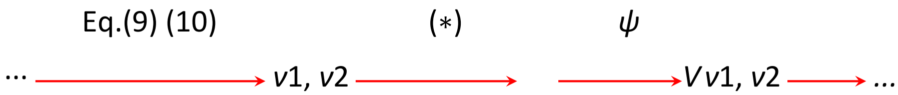
This iterative procedure circumvents many of the challenges resulting from solving the system of EHJB equations (17)–(23) directly. More specifically, we no longer deal with the high dimensionality of the function $f$, and the iterative procedure at each time point $t_n$ has now been eliminated. Also, the determination of an equilibrium stopping rule $\hat{u}$ from Eq. (25) is much simpler than solving the complex Eq. (17). However, since we bypass solving the system of EHJB equations (17)–(23) directly by solving for $v^1$ and $v^2$ (through their PDEs), we need to establish that an equilibrium stopping rule generated by going through **Steps 1–4** is the same as the equilibrium stopping rule resulting from the solution of the system of EHJB equations (17)–(23). The following theorem establishes this equivalence.

**Theorem 2.** *Let u^ be an equilibrium stopping rule resulting from the system of EHJB equations* (17)–(23)*. For any* (*t,F,m*) ∈ [0,*T*]×O¯ *s.t. u^* = 0*, let* $v^p$ *(p* = 1,2*) satisfy Eq.* (9) *with boundary conditions* (10) *with ζ* = 0*. For any* (*t,F,m*) ∈ [0,*T*] × O¯ *s.t. u^* = 1*, let* $v^1(t,F,m) = \psi(t,F)$ *and* $v^2(t,F,m) = \psi^2(t,F)$*. Then*

$$\begin{cases} V(t,F,m) := \nu^1(t,F,m;\tau^{\hat{u}}) - \frac{\gamma(F)}{2}\left[\nu^2(t,F,m;\tau^{\hat{u}}) - \left(\nu^1(t,F,m;\tau^{\hat{u}})\right)^2\right] \\ f(t,F,m,y) := \nu^1(t,F,m;\tau^{\hat{u}}) - \frac{\gamma(F)}{2}\nu^2(t,F,m;\tau^{\hat{u}}) \\ g(t,F,m) := \nu^1(t,F,m;\tau^{\hat{u}}) \end{cases} \tag{27}$$

*satisfy the system of EHJB equations* (17)–(23)*.*

The proof of Theorem 2 is in Appendix A.4. Upon the establishment of Theorem 2, we can determine an equilibrium stopping rule numerically by going through **Steps 1–4**. A detailed algorithmic procedure is as follows:




Partition [0,*T*] into *N* equal time intervals such that $\Delta^t = \frac{T}{N}$ and $t_n = T - n\Delta t$ for *n* = 0,1,...,*N*. Partition [0,1] into *K* equal intervals such that $\Delta^z = \frac{1}{K}$ and $z_i = i\Delta z$ for *i* = 0,1,...,*K*. Truncate the domain of *m* at $m_{\max}$ and partition [0,$m_{\max}$] into *M* equal intervals such that $\Delta^m = \frac{m_{\max}}{M}$ and $m_j = j\Delta_m$ for *j* = 0,1,...,*M*. Define $V_{i,j}^n := V(t_n,z_i,m_j)$, $(v_u^1)^n_{i,j} := v^1(t_n,z_i,m_j;\tau^u)$, $(v_u^2)^n_{i,j} := v^2(t_n,z_i,m_j;\tau^u)$, and $\psi_{i,j}^n := \psi(t_n,z_i,m_j)$ for *n* = 1,...,*N*, *i* = 0,...,*K*, *j* = 0,...,*M*, and *u* ∈ {0,1}. Moreover, if *u* = 1, we have $(\nu_u^1)^n_{i,j} = (\nu_1^1)^n_{i,j} = \psi^n_{i,j}$ and $(\nu_u^2)^n_{i,j} = (\nu_1^2)^n_{i,j} = (\psi^n_{i,j})^2$.

**Algorithm 1 Equilibrium Stopping Rule**

**Figure 6: A Diagram Depicting the Propagation of the Procedure for Algorithm 1**

*Figure 6 gives a diagram depicting the propagation of procedures to better illustrate Algorithm 1.*

```
Set V^0_{i,j} = z_i m_j ∨ G, (ν^1)^0_{i,j} = z_i m_j ∨ G, and (ν^2)^0_{i,j} = (z_i m_j ∨ G)^2 for i = 0,...,K and j = 0,...,M.
for n = 0,...,N − 1 do
    Obtain (ν^1_0)^{n+1}_{i,j} and (ν^2_0)^{n+1}_{i,j} for i = 0,...,K and j = 0,...,M using finite difference method to solve
        Eq. (9) and (10) with ζ = 0 and p = 1,2 for ν^1 and ν^2, respectively.
    Determine û by V^{n+1}_{i,j} = max_{u∈{0,1}} [(ν^1_u)^{n+1}_{i,j} − γ(z_i m_j ∨ G)/2 ((ν^2_u)^{n+1}_{i,j} − ((ν^1_u)^{n+1}_{i,j})^2)] (∗).
    if û = 0 then
        (ν^1_û)^{n+1}_{i,j} = (ν^1_0)^{n+1}_{i,j} and (ν^2_û)^{n+1}_{i,j} = (ν^2_0)^{n+1}_{i,j}
    else
        (ν^1_û)^{n+1}_{i,j} = ^{n+1}_{i,j} and (ν^2_û)^{n+1}_{i,j} = (ψ^{n+1}_{i,j})^2.
    end if
end for
```

### 6.3 Numerical Experiments

In what follows, we conduct some numerical analyses of a sophisticated PH (Type 3) welfare under the HWM fee structure. For completeness, we also present the optimal surrender regions of the PH. For comparative purposes, the welfare and optimal surrender regions of the Type 3 PH under the constant and state-dependent fee structures are also presented.

Table 6 summarizes the welfare of sophisticated PHs (Type 3) with different risk aversion levels for the time horizons $T = 10$ and $T = 25$. Consistent with the results for the Type 1 and Type 2 PHs, the HWM fee structure can be seen to increase a Type 3 PH's welfare. More specifically, the HWM fee structure benefits Type 3 PHs who are more risk-averse (with risk aversion levels $\gamma = 1.2$ and 1.8). However, for a Type 3 PH who is less risk-averse, the state-dependent fee structure is favourable among the fee structures considered. We also observe that the HWM fee structure is more advantageous for a Type 3 PH when the VA contract has a shorter maturity.

**Table 6: Welfare of the Sophisticated PH**

*Table 6 summarizes the sophisticated PHs' welfare with three risk aversion levels over two time horizons under different fee structures. The welfare is evaluated by the MV value function V in Eq.* (15) *at the point* $(t,F,m) = (0,F_0,F_0)$. *The highest welfare is highlighted in bold.*

| | $T = 10$ | | | $T = 25$ | | |
|---|---|---|---|---|---|---|
| Fee structure | $\gamma = 0.6$ | $\gamma = 1.2$ | $\gamma = 1.8$ | $\gamma = 0.6$ | $\gamma = 1.2$ | $\gamma = 1.8$ |
| Constant | 155.71 | 137.34 | 118.99 | 204.31 | 166.69 | 129.13 |
| $\alpha = 0$ | **159.66** | 137.49 | 115.34 | **206.79** | 167.75 | 128.77 |
| $\alpha = 0.05$ | 158.67 | 138.03 | 117.40 | 206.19 | **168.08** | **130.01** |
| $\alpha = 0.2$ | 154.99 | 138.42 | 121.88 | 199.78 | 164.21 | 128.68 |
| $\alpha = 0.5$ | 149.33 | **138.47** | **127.71** | 188.32 | 158.33 | 128.39 |

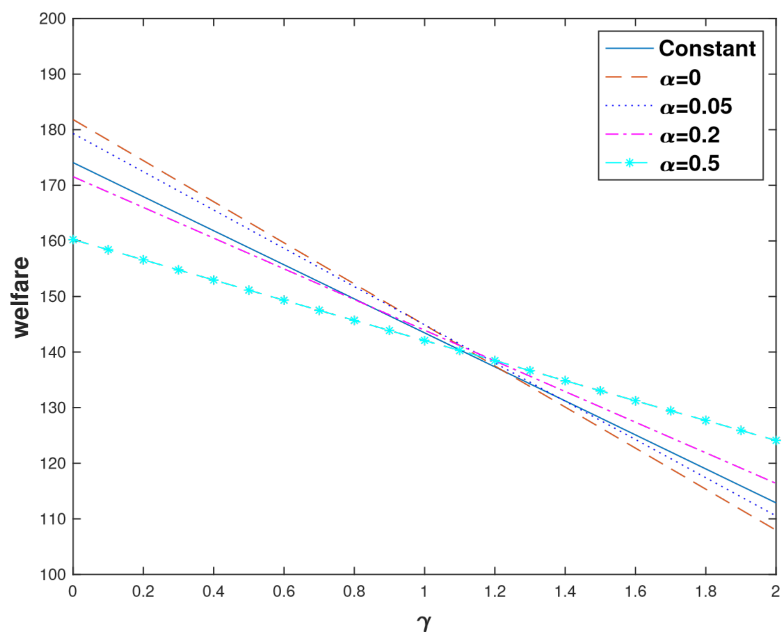
This observation is also in line with that of a Type 2 PH with a large surrender intensity ($\eta = 10$). Indeed, for $T = 10$, increasing the HWM fee $\alpha$ also increases a Type 3 PH's welfare for both $\gamma = 1.2$ and 1.8, while for $T = 25$ the PH's welfare seems to peak earlier as the *highest welfare* is achieved with a HWM fee $\alpha = 0.05$ among the set of fee structures considered.

Similar to the analysis in Sections 4 and 5, we investigate the sensitivity of a group of Type 3 PHs' welfare with respect to risk aversion level $\gamma$. Figure 7 depicts the value of a sophisticated PH welfare across a range of different risk aversion levels for different fee structures when $T = 10$. As can be seen from this figure, the results are consistent with those from Figures 4 and 5. We observe that a higher HWM fee $\alpha$ leads to less sensitive (flatter welfare line) Type 3 PHs' welfare with respect to the risk aversion level $\gamma$. Therefore, we can also conclude that the HWM fee structure is more robust and consequentially more marketable for a group of Type 3 PHs with heterogeneous risk preferences.

For completeness, we examine a Type 3 PH's optimal surrender behaviour under the HWM fee structure. For comparative purposes, the PH's surrender behaviour under the constant and state-dependent fee structures is also included. For the following analysis, we assume $T = 25$ and $\gamma = 1.8$. Figure 8 presents the optimal surrender regions of a sophisticated PH (Type 3) under different fee structures at time $t = 12$. Note that we present an enlarged version of the original surrender regions since the original ones are quite small. There are a few observations worthy of mention. Firstly, compared to surrender regions of a risk-neutral-pricing PH, surrender regions of a Type 3 PH are significantly diminished under all fee structures. As mentioned before, there is discordance between the empirical PH behaviour and the risk-neutral-pricing behaviour.

**Figure 7: The Welfare of Sophisticated PHs across Different Risk Aversions**

*Figure 7 depicts the change of sophisticated PHs' welfare across different risk aversion levels under different fee structures.*



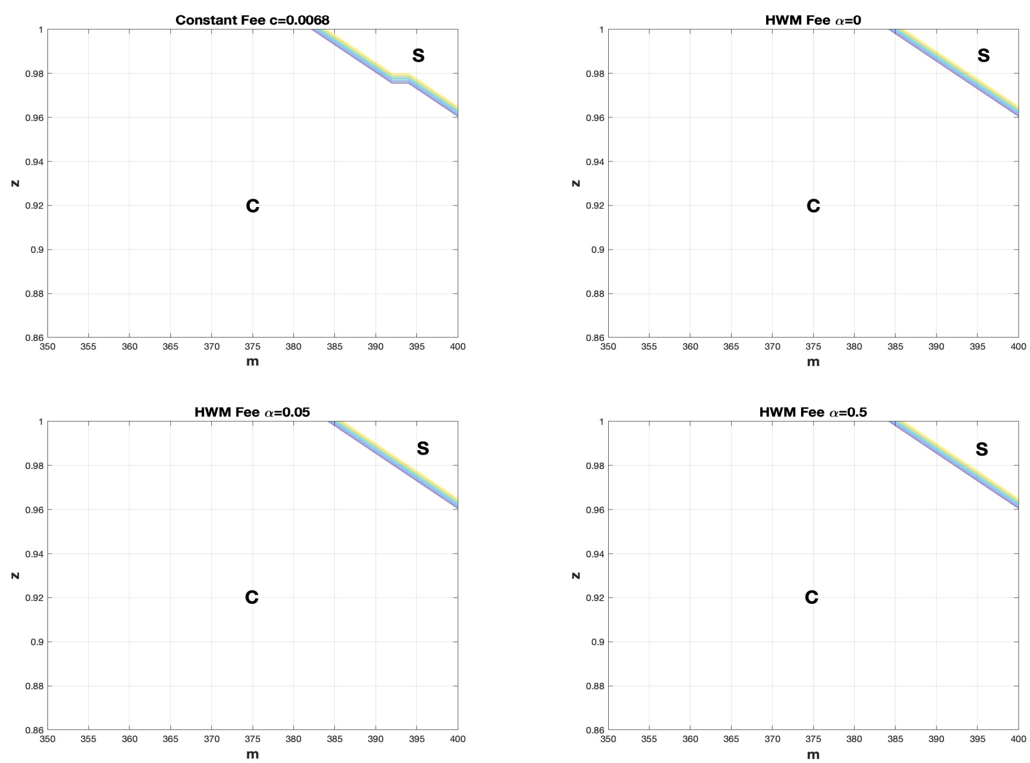

Specifically, PHs exercise their surrender rights far less frequently than the risk-neutral-pricing behaviour suggests. Although this market phenomenon can be explained by PHs' sub-optimal behaviour or market friction (see Moenig and Bauer, 2015), we deem that PHs' behaviour is more closely aligned with that of an MV-maximizing PH.

Further, the optimal surrender regions in Figure 8 are located in the upper right corner of each panel. This observation indicates that the Type 3 PH's surrender behaviour is certainly different from that of a risk-neutral-pricing PH, especially under the state-dependent and HWM fee structures whose optimal surrender regions, as shown before, normally entail a banded-shape zone when the investment account is lower than the threshold $\vartheta$.

Finally, for a Type 3 PH the difference in the surrender regions under different fee structures is somewhat negligible. In other words, changing fee structures has little impact on the surrender behaviour of a sophisticated PH with MV-maximizing strategy. This highlights an advantage of the MV-maximizing strategy for the insurer; that is, if PHs follow an MV-maximizing strategy, the insurer should not anticipate material changes in PHs' behaviour under different fee structures, which is expected to simplify the insurer's risk management activities related to a large portfolio of VAs.

**Figure 8: Optimal Surrender Regions for a Sophisticated PH**

*Figure 8 presents the optimal surrender region of a sophisticated (Type 3) PH at time* t *= 12. The surrender regions presented are an enlarged version of the original surrender regions.*



# 7 Conclusion

In this paper, we introduce a novel HWM fee structure, show its merits to control the variance of the VA payout by stabilizing the investment account, and discuss its implications from the standpoint of both the insurer and the PH.

Under the conventional risk-neutral pricing framework to find a pair of fair fees for the VA, we later conduct a comprehensive analysis of a PH welfare whose risk preference is quantified by an MV strategy. Among other benefits, it was demonstrated that the VA with the novel HWM fee structure can, in some cases, help the PH achieve a higher level of welfare.

Future research should explore the impact of the HWM fee structure under various guarantees; for instance, withdrawal-type benefits, or under multiple market models such as a stochastic volatility model.

Dynamic hedging of a VA policy under the HWM fee structure is also worth investigating.

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# A Some Proofs

## A.1 Verification Theorem for the Pricing PDE

In Section 2.4, the pricing PDE (3) with boundary conditions (4) is presented. Note that Eq. (3) corresponds to the PDE that the value function satisfies over the continuation region $C_t$. Over the region $[0,T] \times O^-$, the value function satisfies the following variational inequality

$$\min\{-V_t - LV - \lambda_{x+t} \max\{F,G\}, V - \psi\} = 0. \quad (28)$$

We prove the following verification theorem for Eq. (28) with boundary conditions (4).

**Theorem 3.** *Suppose that there exists a function* $v \in C^{1,2,1}([0,T] \times O^-)$ *satisfying the quadratic growth condition*[7] *and define* $\tau^* := \inf\{s > t : v(s,X_s) = \psi(s,X_s)\}$. *If*

- *for any* $(t,x) \in [0,T] \times O^-$, *v satisfies the variational inequality* (28), *and*
- *v satisfies the boundary conditions* (4),

*then* $v = V$ *and* $\tau^*$ *is an optimal stopping time.*

Note that in Theorem 3, we let $X_t = (F_t,M_t)$ for $t \in [0,T]$ and $x = (F,m)$ for ease of notation.

*Proof.* By the boundary conditions (4), we establish that $\tau^* \in T_{[t,T]}$ since $v(T,x) = \psi(T,F_T) = \max(F,G_T)$. Define a sequence of stopping times

$$\tau_t^n = \tau^* \wedge \inf\{s > t : |X_s^{t,x} - x| > n\}.$$

Also, $-v_t - \mathcal{L}v - \lambda_{x+t} \max(F,G) = 0$ on $[t, \tau_t^n)$ since $[t, \tau_t^n) \subset [t, \tau^*)$ by definition. Therefore, by Itô's formula and letting $\xi_s = \lambda_{x+s} + r$ for $s \in [0,T]$, we establish that

$$\begin{aligned}
&e^{-\int_0^{\tau_t^n} \xi_u du} v(\tau_t^n, X_{\tau_t^n}^{t,x}) - e^{-\int_0^t \xi_u du} v(t,x) \\
&= \int_0^{\tau_t^n} e^{-\int_0^s \xi_u du} (v_t + \mathcal{L}v)(s, X_s^{t,x}) ds + \int_0^{\tau_t^n} e^{-\int_0^s \xi_u du} v_F \sigma F_s dW_s^{\mathbb{Q}} + \int_0^{\tau_t^n} e^{-\int_0^s \xi_u du} (v_m - v_F \alpha \mathbb{1}_{\{F_s \geq \theta\}}) dM_s \\
&= \int_0^{\tau_t^n} e^{-\int_0^s \xi_u du} (v_t + \mathcal{L}v)(s, X_s^{t,x}) ds + \int_0^{\tau_t^n} e^{-\int_0^s \xi_u du} v_F \sigma F_s dW_s^{\mathbb{Q}}.
\end{aligned}$$

Note that the second equality is made possible by the fact that $(v_m - v_F\alpha 1_{Ft\geq\vartheta})dM_t = 0$ a.s., which holds since either $dM_t = 0$ for $F_t < M_t$ or $v_m - v_F\alpha 1_{Ft\geq\vartheta} = 0$ for $F_t = M_t$ by boundary conditions (4). Taking the conditional expectation $E^Q_{t,x}[\cdot]$ on both sides of this equality and then rearranging the resulting equation by further making use of Eq. (3), we get

$$v(t,x) = \mathbb{E}_{t,x}^{\mathbb{Q}} \left[ e^{-\int_t^{\tau_t^n} \xi_u du} v(\tau_t^n, X_{\tau_t^n}^{t,x}) + \int_t^{\tau_t^n} e^{-\int_t^s \xi_u du} \lambda_{x+s} \max\{F_s, G_s\} ds \right].$$

[7] There is a constant $C$ such that $|v(t,x)| \leq C(1 + |x|^2)$ where $x = (F,m)$ and $(t,x) \in [0,T) \times O^-$.

Since $\tau_t^n \to \tau^*$ as $n \to \infty$, by dominated convergence theorem and the quadratic growth condition, it follows that

$$v(t,x) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left[e^{-\int_t^{\tau^*}\xi_u du}\psi(\tau^*, F_{\tau^*}) + \int_t^{\tau^*} e^{-\int_t^s \xi_u du}\lambda_{x+s}\max\{F_s, G_s\}ds\right].$$

For any $\tau \in \mathrm{T}_{[t,T]}$, we have $-v_t - \mathrm{L}v - \lambda_{x+t}\max\{F,G\} \geq 0$ on $[t,\tau)$ since $V \geq \psi$ by the definition of $\tau^*$.

Therefore,

$$v(t,x) \geq \mathbb{E}^{\mathbb{Q}}_{t,x}\left[e^{-\int_t^{\tau}\xi_u du}\psi(\tau, F_{\tau}) + \int_t^{\tau} e^{-\int_t^s \xi_u du}\lambda_{x+s}\max\{F_s, G_s\}ds\right]$$

for any $\tau \in \mathrm{T}_{[t,T]}$. Hence, we have $v = V$ and $\tau^*$ is an optimal stopping time. □

## A.2 Proof of Proposition 1

*Proof.* Suppose that the stopping time $\tau$ follows an exponential distribution with parameter $\eta$, independently of the future lifetime of the PH $\rho$ and the performance of the investment account $\{F_t\}_{t\in[0,T]}$. The first moment of the VA payout is

*v*1(*t,F,m*) = E*t,F,m* he−*ζ*(*τ*∧*ρ*∧*T*−*t*)*H*(*Fτ*∧*ρ*∧*T*)i.

We consider three cases, namely $\tau \wedge \rho > T$, $\tau < \rho \wedge T$, and $\rho < \tau \wedge T$. Also, for simpler notation, we write E*t,F,m*[·] = E*t*[·] and $v^1(t,F,m) = v^1$. It follows that

$$\nu^1 = \mathbb{E}_t\left[\underbrace{e^{-\zeta(T-t)}\max(F_T,G_T)\mathbb{1}_{\{\tau\wedge\rho>T\}}}_{\text{Term }\mathbf{A}} + \underbrace{e^{-\zeta(\tau-t)}(1-\kappa_\tau)F_\tau\mathbb{1}_{\{\tau<\rho\wedge T\}}}_{\text{Term }\mathbf{B}} + \underbrace{e^{-\zeta(\rho-t)}\max(F_\rho,G_\rho)\mathbb{1}_{\{\rho<\tau\wedge T\}}}_{\text{Term }\mathbf{C}}\right],$$

where

- for Term **A**:

$$\begin{aligned}\mathbb{E}_t[\mathbf{A}] &= \mathbb{E}_t\left[\mathbb{E}\left[e^{-\zeta(T-t)}\max(F_T,G_T)\mathbb{1}_{\{\tau\wedge\rho>T\}}\middle|F_T,\tau\wedge\rho>t\right]\right]\\ &= \mathbb{E}_t\left[e^{-\zeta(T-t)}\max(F_T,G_T)\frac{\mathbb{P}(\tau\wedge\rho>T)}{\mathbb{P}(\tau\wedge\rho>t)}\right]\\ &= \mathbb{E}_t\left[e^{-\int_t^T(\zeta+\lambda_{x+s}+\eta)ds}\max(F_T,G_T)\right],\end{aligned}$$

- for Term **B**:

$$\begin{aligned}\mathbb{E}_t[\mathbf{B}] &= \mathbb{E}_t\left\{\mathbb{E}\left[e^{-\zeta(\tau-t)}(1-\kappa_\tau)F_\tau\mathbb{1}_{\{\tau<\rho\wedge T\}}|\tau\wedge\rho>t\right]\right\}\\ &= \mathbb{E}_t\left[\int_t^T\left(\int_s^T {}_{u-t}p_{x+t}\lambda_{x+u}du + {}_{T-t}p_{x+t}\right)(1-\kappa_s)F_s\eta e^{-(\zeta+\eta)(s-t)}ds\right]\\ &= \mathbb{E}_t\left[\int_t^T e^{-\int_t^s(\zeta+\lambda_{x+v}+\eta)dv}\eta(1-\kappa_s)F_s ds\right],\end{aligned}$$

- and for Term **C**:

$$\begin{aligned}\mathbb{E}_t[\mathbf{C}] &= \mathbb{E}_t\left[\mathbb{E}\left[e^{-\zeta(\rho-t)}\max(F_\rho,G_\rho)\mathbb{1}_{\{\rho<\tau\wedge T\}}|\tau\wedge\rho>t\right]\right]\\ &= \mathbb{E}_t\left[\int_t^T\left(\int_u^T\eta e^{-\eta(s-t)}ds + e^{-\eta(T-t)}\right)\lambda_{x+u}e^{-\int_t^u(\zeta+\lambda_v)dv}\max(F_u,G_u)du\right]\\ &= \mathbb{E}_t\left[\int_t^T e^{-\int_t^s(\zeta+\lambda_{x+v}+\eta)dv}\lambda_{x+s}\max(F_s,G_s)ds\right].\end{aligned}$$

Combining Terms **A**, **B** and **C**, we obtain

$$\nu^1 = \mathbb{E}_t\left[e^{-\int_t^T\delta_s^1 ds}\max(F_T,G_T) + \int_t^T e^{-\int_t^s\delta_u^1 du}\left[\eta(1-\kappa_s)F_s + \lambda_{x+s}\max(F_s,G_s)\right]ds\right].$$

The expression for $v^2(t,F,m)$ is derived similarly. □

### A.3 Proof of Theorem 1

To prove Theorem 1, we first introduce some notation and results for brevity of illustration.

- $\mathbb{E}_{t,F,m}[\cdot] = \mathbb{E}_t[\cdot]$ and $\mathrm{Var}_{t,F,m}[\cdot] = \mathrm{Var}_t[\cdot]$.
- $\mathcal{L}h(t,F,m) := h_t + (\mu - c\mathbb{1}_{\{F<\theta\}})Fh_F + \frac{1}{2}\sigma^2F^2h_{FF}$ for a function $h$ s.t. $h \in C^{1,2,1}([0,T]\times\bar{\mathcal{O}})$.
- $\mathcal{L}f(t,F,m,y) := f_t + (\mu - c\mathbb{1}_{\{F<\theta\}})(Ff_F + yf_y) + \frac{1}{2}\sigma^2(F^2f_{FF} + y^2f_{yy}) + f_{Fy}Fy\sigma^2$ for a function $f$ s.t. $f \in C^{1,2,1,2}([0,T]\times\bar{\mathcal{O}}\times[0,m])$.
- $\mathbb{E}_t\left[h(t+\varepsilon, F_{t+\varepsilon}, M_{t+\varepsilon})\right] = h(t,F,m) + \varepsilon\mathcal{L}h(t,F,m) + \mathbb{E}_t\left[\int_t^{t+\varepsilon}(h_m - \alpha\mathbb{1}_{\{F\geq\theta\}}h_F)dM_s\right] + o(\varepsilon)$.
- $\mathbb{E}_t\left[f(t+\varepsilon, F_{t+\varepsilon}, M_{t+\varepsilon}, y_{t+\varepsilon})\right] = f(t,F,m,y) + \varepsilon\mathcal{L}f(t,F,m,y) + \mathbb{E}_t\left[\int_t^{t+\varepsilon} f_m - (f_F + f_y)\alpha\mathbb{1}_{\{F\geq\theta\}}dM_s\right] + o(\varepsilon)$.

*Proof.* The proof of Theorem 1 can be divided into two parts. In the first part, we prove that *V*, *g*, and *f* are indeed the solution of their probabilistic interpretation (24) under the stopping rule $\hat{u}$. In the second part, we prove that $\hat{u}$ is an equilibrium stopping rule. Firstly, we prove that

$$g(t,F,m) = \mathbb{E}_t\left[H(F_{\tau^{\hat{u}}\wedge\rho})\big|\rho > t\right]$$
$$= \mathbb{E}_t\left[e^{-\int_t^{\tau^{\hat{u}}}\lambda_{x+u}du}\psi(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}) + \int_t^{\tau^{\hat{u}}} e^{-\int_t^s\lambda_{x+u}du}\lambda_{x+s}\max(F_s,G_s)ds\right],$$

where

$$H(F_{\tau^{\hat{u}}\wedge\rho}) = \begin{cases}\max(F_\rho, G_\rho), & \text{if } \rho < \tau^{\hat{u}},\\ \psi(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}), & \text{if } \rho \geq \tau^{\hat{u}}.\end{cases}$$

Suppose that $g \in C^{1,2,1}([0,T]\times\hat{\mathcal{O}})$ satisfies Eq. (22) with boundary conditions (23), and define a sequence of stopping times

$$\tau_t^n := \tau^{\hat{u}} \wedge \inf\left\{s > t : |F_s^{t,F} - F| > n\right\}.$$

Applying Itô's formula to $e^{-\int_0^s\lambda_u du}g(s,F_s,M_s)$, we get

$$e^{-\int_t^{\tau_t^n}\lambda_{x+u}du}g(\tau_t^n, F_{\tau_t^n}, M_{\tau_t^n}) - g(t,F,m)$$
$$= \int_t^{\tau_t^n} e^{-\int_t^s\lambda_{x+u}du}(\mathcal{L}g - \lambda_{x+s}g)ds + \int_t^{\tau_t^n} e^{-\int_t^s\lambda_{x+u}du}(g_m - \alpha\mathbb{1}_{\{F_s>\theta\}}g_F)dM_s$$
$$+ \int_t^{\tau_t^n} e^{-\int_t^s\lambda_{x+u}du}\sigma F_s g_F dW_s.$$

Taking E*t*[·] on both sides of the equation and invoking Eq. (22) with boundary conditions (23), we obtain

$$g(t,F,m)=\mathbb{E}_t\left[e^{-\int_t^{\tau_t^n}\lambda_{x+u}du}g(\tau_t^n,F_{\tau_t^n},M_{\tau_t^n})+\int_t^{\tau_t^n}e^{-\int_t^s\lambda_{x+u}du}\lambda_{x+s}\max(F_s,G_s)ds\right].$$

Letting $n\to\infty$ gives $\tau_t^n\to\tau^{\hat{u}}$, which further leads to

$$\begin{aligned}g(t,F,m)&=\mathbb{E}_t\left[e^{-\int_t^{\tau^{\hat{u}}}\lambda_{x+u}du}g(\tau^{\hat{u}},F_{\tau^{\hat{u}}},M_{\tau^{\hat{u}}})+\int_t^{\tau^{\hat{u}}}e^{-\int_t^s\lambda_{x+u}du}\lambda_{x+s}\max(F_s,G_s)ds\right]\\&=\mathbb{E}_t\left[e^{-\int_t^{\tau^{\hat{u}}}\lambda_{x+u}du}\psi(\tau^{\hat{u}},F_{\tau^{\hat{u}}})+\int_t^{\tau^{\hat{u}}}e^{-\int_t^s\lambda_{x+u}du}\lambda_{x+s}\max(F_s,G_s)ds\right]\\&=\mathbb{E}_t\left[H(F_{\tau^{\hat{u}}\wedge\rho})|\rho>t\right],\end{aligned}$$

since

$$\tau^{\hat{u}}=\inf\{s\geq t:\hat{u}(t,F,m)=1\}.$$

Define $f^y(t,F,m):=f(t,F,m,y)$ for a fixed $y$ and similarly, we apply Itô's formula to $e^{-\int_0^s\lambda_{x+u}du}f^y(s,F_s,M_s)$ and obtain

$$\begin{aligned}&e^{-\int_t^{\tau_t^n}\lambda_{x+u}du}f^y(\tau_t^n,F_{\tau_t^n},M_{\tau_t^n})-f^y(t,F,m)\\&=\int_t^{\tau_t^n}e^{-\int_t^s\lambda_{x+u}du}(\mathcal{L}f^y-\lambda_{x+s}f^y)ds+\int_t^{\tau_t^n}e^{-\int_t^s\lambda_{x+u}du}(f_m^y-\alpha\mathbb{1}_{\{F_s>\theta\}}f_F^y)dM_s\\&+\int_t^{\tau_t^n}e^{-\int_t^s\lambda_{x+u}du}\sigma F_sf_F^ydW_s.\end{aligned}$$

Taking E*t*[·] on both sides of the equation and invoking Eq. (20) with boundary conditions (21) results in

$$\begin{aligned}f^y(t,F,m)=\mathbb{E}_t\Bigg[&e^{-\int_t^{\tau_t^n}\lambda_{x+u}du}f^y(\tau_t^n,F_{\tau_t^n},M_{\tau_t^n})\\&+\int_t^{\tau_t^n}e^{-\int_t^s\lambda_{x+u}du}\lambda_{x+s}\left(\max(F_s,G_s)-\frac{\gamma(y)}{2}\max(F_s,G_s)^2\right)\Bigg].\end{aligned}$$

By letting $n \to \infty$, we have $\tau_t^n \to \tau^{\hat{u}}$, which in turn leads to

$$f^y(t,F,m) = \mathbb{E}_t\Bigg[e^{-\int_t^{\tau^{\hat{u}}} \lambda_{x+u}du} f^y(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}, M_{\tau^{\hat{u}}}) + \int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du}\lambda_{x+s}\left(\max(F_s,G_s)ds - \frac{\gamma(y)}{2}\max(F_s,G_s)^2\right)\Bigg]$$
$$= \mathbb{E}_t\left[e^{-\int_t^{\tau^{\hat{u}}} \lambda_{x+u}du}\psi(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}) + \int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du}\lambda_{x+s}\max(F_s,G_s)ds\right]$$
$$- \frac{\gamma(y)}{2}\mathbb{E}_t\left[e^{-\int_t^{\tau^{\hat{u}}} \lambda_{x+u}du}\psi^2(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}) + \int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du}\lambda_{x+s}\max(F_s,G_s)^2ds\right]$$
$$= \mathbb{E}_t\left[H(F_{\tau^{\hat{u}}\wedge\rho})\big|\rho > t\right] - \frac{\gamma(y)}{2}\mathbb{E}_t\left[H^2(F_{\tau^{\hat{u}}\wedge\rho})\big|\rho > t\right].$$

To verify the probabilistic interpretation of $V$, we need to show that

$$V(t,F,m) = \frac{\gamma(x)}{2}g^2(t,F,m) + f(t,F,m,F).$$

Since $V(T,F,m) = \psi(T,F)$, we have $\tau^{\hat{u}} \in \mathrm{T}_{[t,T]}$. Applying Itô's formula to $e^{-\int_0^s \lambda_u du}V(s,F_s,M_s)$, we get

$$e^{-\int_t^{\tau_t^n} \lambda_{x+u}du}V(\tau_t^n, F_{\tau_t^n}, M_{\tau_t^n}) - V(t,F,m)$$
$$= \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u}du}(\mathcal{L}V - \lambda_{x+s}V)ds + \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u}du}(V_m - \alpha\mathbb{1}_{\{F_s>\theta\}}V_F)dM_s$$
$$+ \int_t^{\tau_t^n} e^{-\int_t^s \lambda_{x+u}du}\sigma F_s V_F dW_s.$$

Since $[t,\tau_t^n) \subset [t,\tau^{\hat{u}})$, we deduce that $V > \psi$ on $[t,\tau_t^n)$. Using Eq. (17) for $V > \psi$, taking $E_t[\cdot]$ on both sides of the equation and letting $n \to \infty$, we obtain

$$V(t,F,m) = \mathbb{E}_t\left[e^{-\int_t^{\tau^{\hat{u}}} \lambda_{x+u}du}V(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}, M_{\tau^{\hat{u}}})\right] - \mathbb{E}_t\Bigg\{\int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du}\Bigg[(\mu - c\mathbb{1}_{\{F_s<\theta\}})F_s\left(f_y + \frac{\gamma_F}{2}g^2\right)$$
$$+ \frac{1}{2}\sigma^2F_s^2\left(f_{yy} + 2f_{Fy} + \frac{\gamma_{FF}}{2}g^2 + \gamma(F_s)g_F^2 + 2\gamma_F g g_F\right) + \frac{\gamma(F_s)}{2}\lambda_{x+s}\left(g - \max(F_s,G_s)\right)^2$$
$$- \lambda_{x+s}\max(F_s,G_s)\Bigg]ds\Bigg\} - \mathbb{E}_t\left[\int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du}\left(V_m - \alpha\mathbb{1}_{\{F_s\geq\theta V_F\}}\right)dM_s\right].$$

Now, applying Itô's formula and conducting operations as above on $e^{-\int_0^s \lambda_{x+u}du}f(s,F_s,M_s,F_s)$, we deduce

$$f(t,F,m,F) = \mathbb{E}_t\left[e^{-\int_t^{\tau^{\hat{u}}} \lambda_{x+u}du}f(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}, M_{\tau^{\hat{u}}}, F_{\tau^{\hat{u}}})\right] - \mathbb{E}_t\Bigg\{\int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u}du}\Bigg[-\lambda_{x+s}\max(F_s,G_s)$$
$$+ \lambda_{x+s}\frac{\gamma(F_s)}{2}\max(F_s,G_s)^2 + f_yF_s(\mu - \mathbb{1}_{\{F_s<\theta\}}) + \frac{1}{2}F_s^2\sigma^2 f_{yy} + F_s^2\sigma^2 f_{Fy}\Bigg]ds\Bigg\}$$
$$+ \mathbb{E}_t\left(\int_t^{\tau^{\hat{u}}} f_y\alpha\mathbb{1}_{\{F_s\geq\theta\}}dM_s\right).$$

Again, applying Itô's formula and conducting operations as above on $e^{-\int_0^s \lambda_{x+u} du} \frac{\gamma(F_s)}{2} g(s, F_s, M_s)$), we have

$$\frac{\gamma(F)}{2} g^2(t,F,m) = \mathbb{E}_t \left[ e^{-\int_t^{\tau^{\hat{u}}} \lambda_{x+u} du} \frac{\gamma(F_{\tau^{\hat{u}}})}{2} g\left(\tau^{\hat{u}}, F_{\tau^{\hat{u}}}, M_{\tau^{\hat{u}}}\right) \right] - \mathbb{E}_t \Bigg\{ \int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u} du} \Big[ -\gamma(F_s) g \lambda_{x+s} \max(F_s, G_s) + \frac{\gamma(F_s)}{2} \lambda_{x+s} g^2 + \frac{\gamma(F_s)}{2} g_F^2 \sigma^2 F_s^2 + \frac{g_F^2}{2} \gamma_F (\mu - c \mathbb{1}_{\{F_s < \theta\}}) F_s + \frac{1}{4} g^2 \gamma_{FF} \sigma^2 F_s^2 + \gamma_F \sigma^2 F_s^2 g g_F) \Big] ds \Bigg\} + \mathbb{E}_t \left[ \int_t^{\tau^{\hat{u}}} e^{-\int_t^s \lambda_{x+u} du} \frac{\gamma_F}{2} g^2 \alpha \mathbb{1}_{\{F_s \geq \theta\}} dM_s \right].$$

Combining the expressions for *f*(*t,F,m,F*) and $\frac{\gamma(F)}{2} g^2(t,F,m)$), it is easy to establish that

$$V(t,F,m) = \frac{\gamma(x)}{2} g^2(t,F,m) + f(t,F,m,F).$$

Now we prove that the stopping rule *u*ˆ defined in Eq. (1) is an equilibrium stopping rule. Recall that we define a stopping rule $u^\varepsilon$ such that

$$u^\varepsilon(s,y,z) = \begin{cases} u \in \{0,1\}, & \text{for } t \leq s < (t+\varepsilon) \wedge \rho \\ \hat{u}(s,y,z), & \text{for } (t+\varepsilon) \wedge \rho \leq s < T \wedge \rho. \end{cases}$$

If *u* = 1, then J(*t,F,m*;$\tau^{u\varepsilon}$) = $\psi$(*t,F*) ≤ J(*t,F,m*;$\tau^{u\hat{}}$) = *V* (*t,F,m*). Therefore, *u*ˆ automatically satisfies the definition of an equilibrium stopping rule. If *u* = 0, we have

$$\begin{aligned}
\mathcal{J}(t,F,m;\tau^{\boldsymbol{u}^\varepsilon}) &= \mathbb{E}_t \left[ H(F_{\tau^{\boldsymbol{u}^\varepsilon} \wedge \rho}) \middle| \rho > t \right] - \frac{\gamma(F)}{2} \mathrm{Var}_t \left[ H(F_{\tau^{\boldsymbol{u}^\varepsilon} \wedge \rho}) \middle| \rho > t \right] \\
&= \mathbb{E}_t \left[ H(F_{\tau^{\boldsymbol{u}^\varepsilon} \wedge \rho}) - \frac{\gamma(F)}{2} H^2(F_{\tau^{\boldsymbol{u}^\varepsilon} \wedge \rho}) \middle| \rho > t \right] + \frac{\gamma(F)}{2} \left\{ \mathbb{E}_t \left[ H(F_{\tau^{\boldsymbol{u}^\varepsilon} \wedge \rho}) \middle| \rho > t \right] \right\}^2 \\
&= e^{-\int_t^{t+\varepsilon} \lambda_{x+u} du} \mathbb{E}_t \left[ H(F_{\tau^{\boldsymbol{u}^\varepsilon} \wedge \rho}) - \frac{\gamma(F)}{2} H^2(F_{\tau^{\boldsymbol{u}^\varepsilon} \wedge \rho}) \middle| \rho > t + \varepsilon \right] \\
&+ \int_t^{t+\varepsilon} e^{-\int_t^s \lambda_{x+u} du} \lambda_{x+s} \mathbb{E}_t \left[ H(F_s) - \frac{\gamma(F)}{2} H^2(F_s)) \right] ds \\
&+ \frac{\gamma(F)}{2} \left\{ e^{-\int_t^{t+\varepsilon} \lambda_{x+u} du} \mathbb{E}_t \left[ H(F_{\tau^{\boldsymbol{u}^\varepsilon} \wedge \rho}) \middle| \rho > t + \varepsilon \right] + \int_t^{t+\varepsilon} e^{-\int_t^s \lambda_{x+u} du} \lambda_{x+s} \mathbb{E}_t \left[ H(F_s) \right] \right\}^2 \\
&= (1 - \lambda_{x+t}\varepsilon) \mathbb{E}_t \left\{ \mathbb{E}_{t+\varepsilon} \left[ H(F_{\tau^{\hat{\boldsymbol{u}}} \wedge \rho}) - \frac{\gamma(F)}{2} H^2(F_{\tau^{\hat{\boldsymbol{u}}} \wedge \rho}) \middle| \rho > t + \varepsilon \right] \right\} + \varepsilon \lambda_{x+t} \left[ H(F) - \frac{\gamma(F)}{2} H^2(F) \right] \\
&+ \frac{\gamma(F)}{2} \left\{ (1 - \lambda_{x+t}\varepsilon) \mathbb{E}_t \left[ \mathbb{E}_{t+\varepsilon} \left( H(F_{\tau^{\hat{\boldsymbol{u}}} \wedge \rho}) \middle| \rho > t + \varepsilon \right) \right] + \lambda_{x+t} \varepsilon H(F) \right\}^2 + o(\varepsilon).
\end{aligned}$$

Also, we have

$$\mathcal{J}(t+\varepsilon, F_{t+\varepsilon}, M_{t+\varepsilon}; \tau^{\hat{\boldsymbol{u}}}) = \mathbb{E}_{t+\varepsilon} \left[ H(F_{\tau^{\hat{\boldsymbol{u}}} \wedge \rho}) - \frac{\gamma(F_{t+\varepsilon})}{2} H^2(F_{\tau^{\hat{\boldsymbol{u}}} \wedge \rho}) \middle| \rho > t + \varepsilon \right] + \frac{\gamma(F_{t+\varepsilon})}{2} \left\{ \mathbb{E}_{t+\varepsilon} \left[ H(F_{\tau^{\hat{\boldsymbol{u}}} \wedge \rho}) \middle| \rho > t + \varepsilon \right] \right\}^2.$$

Taking E$t$[·] on both sides of the equation, we obtain

$$\mathbb{E}_t\left[\mathcal{J}(t+\varepsilon, F_{t+\varepsilon}, M_{t+\varepsilon}; \tau^{\hat{\boldsymbol{u}}})\right] = \mathbb{E}_t\left\{\mathbb{E}_{t+\varepsilon}\left[H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho}) - \frac{\gamma(F_{t+\varepsilon})}{2}H^2(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\Big|\rho > t+\varepsilon\right]\right\} + \mathbb{E}_t\left[\frac{\gamma(F_{t+\varepsilon})}{2}\left\{\mathbb{E}_{t+\varepsilon}\left[H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\Big|\rho > t+\varepsilon\right]\right\}^2\right].$$

Therefore,

$$\begin{aligned}\mathcal{J}(t,F,m;\tau^{\boldsymbol{u}^\varepsilon}) &= (1-\lambda_{x+t}\varepsilon)\mathbb{E}_t\left\{\mathbb{E}_{t+\varepsilon}\left[H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho}) - \frac{\gamma(F)}{2}H^2(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\Big|\rho > t+\varepsilon\right]\right\} + \varepsilon\lambda_{x+t}\left[H(F) - \frac{\gamma(F)}{2}H^2(F)\right] \\ &+ \frac{\gamma(F)}{2}\left\{(1-\lambda_{x+t}\varepsilon)\mathbb{E}_t\left[\mathbb{E}_{t+\varepsilon}\left(H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\Big|\rho > t+\varepsilon\right)\right] + \lambda_{x+t}\varepsilon H(F)\right\}^2 \\ &+ \mathbb{E}_t\left[\mathcal{J}(t+\varepsilon, F_{t+\varepsilon}, M_{t+\varepsilon}; \tau^{\hat{\boldsymbol{u}}})\right] - \mathbb{E}_t\left\{\mathbb{E}_{t+\varepsilon}\left[H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho}) - \frac{\gamma(F_{t+\varepsilon})}{2}H^2(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\Big|\rho > t+\varepsilon\right]\right\} \\ &- \mathbb{E}_t\left[\frac{\gamma(F_{t+\varepsilon})}{2}\left\{\mathbb{E}_{t+\varepsilon}\left[H(F_{\tau^{\hat{\boldsymbol{u}}}\wedge\rho})\Big|\rho > t+\varepsilon\right]\right\}^2\right] + o(\varepsilon) \\ &= (1-\lambda_{x+t}\varepsilon)\mathbb{E}_t\left[f(t+\varepsilon, F_{t+\varepsilon}, M_{t+\varepsilon}, F)\right] + \varepsilon\lambda_{x+t}\left[H(F) - \frac{\gamma(F)}{2}H^2(F)\right] \\ &+ \frac{\gamma(F)}{2}\left\{(1-\lambda_{x+t}\varepsilon)\mathbb{E}_t\left[g(t+\varepsilon, F_{t+\varepsilon}, M_{t+\varepsilon})\right] + \lambda_{x+t}\varepsilon H(F)\right\}^2 + \mathbb{E}_t\left[V(t+\varepsilon, F_{t+\varepsilon}, M_{t+\varepsilon})\right] \\ &- \mathbb{E}_t\left[f(t+\varepsilon, F_{t+\varepsilon}, M_{t+\varepsilon}, F_{t+\varepsilon})\right] - \mathbb{E}_t\left[\frac{\gamma(F_{t+\varepsilon})}{2}g^2(t+\varepsilon, F_{t+\varepsilon}, M_{t+\varepsilon})\right] + o(\varepsilon).\end{aligned}$$

Thus,

$$\begin{aligned}&\mathcal{J}(t,F,m;\tau^{\boldsymbol{u}^\varepsilon}) - \mathcal{J}(t,F,m;\tau^{\hat{\boldsymbol{u}}}) \\ &= \varepsilon\mathcal{L}V(t,F,m) - \varepsilon\lambda_{x+t}f^F(t,F,m) + \varepsilon\mathcal{L}f^F(t,F,m) + \varepsilon\lambda_{x+t}\left[H(F) - \frac{\gamma(F)}{2}H^2(F)\right] \\ &- \varepsilon\gamma(F)\lambda_{x+t}g^2(t,F,m) + \varepsilon\gamma(F)\lambda_{x+t}H(F)g(t,F,m) - \varepsilon\mathcal{L}f(t,F,m,y) - \varepsilon\mathcal{L}\frac{\gamma(F)}{2}g^2(t,F,m) \\ &+ \mathbb{E}_t\left[\int_t^{t+\varepsilon} V_m - \alpha\mathbb{1}_{\{F\geq\theta\}}\left(V_F - f_y - \frac{\gamma_F}{2}g^2(t,F,m)\right)dM_s\right] + o(\varepsilon),\end{aligned}$$

where $f^y(t,F,m) = f(t,F,m,y)$ with the fourth variable $y$ fixed. By rearranging the equation above, we get

$$\begin{aligned}&\mathcal{J}(t,F,m;\tau^{\boldsymbol{u}^\varepsilon}) - \mathcal{J}(t,F,m;\tau^{\hat{\boldsymbol{u}}}) \\ &= \varepsilon\Bigg\{V_t + (\mu - c\mathbb{1}_{\{F\geq\theta\}})FV_F\left(V_F - f_y - \frac{\gamma_F}{2}g^2\right) + \frac{1}{2}\left(V_{FF} - f_{yy} - 2f_{Fy} - \frac{\gamma_{FF}}{2}g^2 - \gamma(F)g_F^2 - 2\gamma_F g g_F\right) \\ &- \frac{\lambda_{x+t}\gamma(F)}{2}(g - \max(F,G))^2 - \lambda_{x+t}V + \lambda_{x+t}\max(F,G)\Bigg\} \\ &+ \mathbb{E}_t\Big[\int_t^{t+\varepsilon} V_m - \alpha\mathbb{1}_{\{F\geq\theta\}}(V_F - f_y - \frac{\gamma_F}{2}g^2(t,F,m))dM_s\Big] + o(\varepsilon).\end{aligned}$$

Invoking Eq. (17) and (18), dividing both sides of the equation by $\varepsilon$, and letting $\varepsilon$ go to 0, we get

$$\lim_{\varepsilon\downarrow 0}\frac{\mathcal{J}(t,F,m;\tau^{\boldsymbol{u}^\varepsilon}) - \mathcal{J}(t,F,m;\tau^{\hat{\boldsymbol{u}}})}{\varepsilon} = \lim_{\varepsilon\downarrow 0}\frac{o(\varepsilon)}{\varepsilon} = 0.$$

This completes the proof.

### A.4 Proof of Theorem 2

*Proof.* Recall that for $\hat{u} = 0$ the first two moments $\nu^p$ ($p = 1,2$) satisfy Eq. (9) with boundary conditions (10) with $\zeta = 0$. Note that from the linear combination (27), we have

$$\begin{cases} V(t,F,m) = \nu^1(t,F,m;\tau^{\hat{u}}) - \frac{\gamma(F)}{2}\left[\nu^2(t,F,m;\tau^{\hat{u}}) - \left(\nu^1(t,F,m;\tau^{\hat{u}})\right)^2\right] \\ f(t,F,m,y) = \nu^1(t,F,m;\tau^{\hat{u}}) - \frac{\gamma(y)}{2}\nu^2(t,F,m;\tau^{\hat{u}}), \\ g(t,F,m) = \nu^1(t,F,m;\tau^{\hat{u}}). \end{cases}.$$

Therefore, the derivatives of $V$ with respect to $t$, $F$, and $m$ can then be expressed in terms of those of $\nu^1$ and $\nu^2$ as follows:

$$\begin{cases} V_t = \nu^1_t - \frac{\gamma(F)}{2}\left(\nu^2_t - 2\nu^1\nu^1_t\right), \\ V_F = \nu^1_F - \frac{\gamma_F}{2}\left(\nu^2 - (\nu^1)^2\right) - \frac{\gamma(F)}{2}\left(\nu^2_F - 2\nu^1\nu^1_F\right), \\ V_{FF} = \nu^1_{FF} - \frac{\gamma_{FF}}{2}\left(\nu^2 - (\nu^1)^2\right) - \gamma_F\left(\nu^2_F - 2\nu^1\nu^1_F\right) - \frac{\gamma(F)}{2}\left(\nu^2_{FF} - 2(\nu^1_F)^2 - 2\nu^1\nu^1_{FF}\right) \\ V_m = \nu^1_m - \frac{\gamma(F)}{2}\left(\nu^2_m - 2\nu^1\nu^1_m\right). \end{cases}.$$

Similarly, for $f$, its derivatives can be expressed as

$$\begin{cases} f_t = \nu^1_t - \frac{\gamma(y)}{2}\nu^2_t, \\ f_F = \nu^1_F - \frac{\gamma(y)}{2}\nu^2_F, \\ f_y\big|_{y=F} = -\frac{\gamma'(F)}{2}\nu^2, \\ f_{FF} = \nu^1_{FF} - \frac{\gamma(y)}{2}\nu^2_{FF} \\ f_{yy}\big|_{y=F} = -\frac{\gamma'(FF)}{2}\nu^2, \\ f_{Fy}\big|_{y=F} = -\frac{\gamma'(F)}{2}\nu^2_F, \\ f_m = \nu^1_m - \frac{\gamma(y)}{2}\nu^2_m. \end{cases}.$$

Firstly, we check if the linear combination (27) solves Eq. (17) with boundary conditions (18). For $\hat{u} = 1$,

$$\begin{aligned} V(t,F,m) &= \nu^1(t,F,m;\tau^{\hat{u}}) - \frac{\gamma(F)}{2}\left[\nu^2(t,F,m;\tau^{\hat{u}}) - \left(\nu^1(t,F,m;\tau^{\hat{u}})\right)^2\right] \\ &= \max(F,G) - \frac{\gamma(F)}{2}\left(\max(F,G)^2 - \max(F,G)^2\right) \\ &= \max(F,G) = \psi(t,F). \end{aligned}$$

For $\hat{u} = 0$, we know that

$$V_t + \left(\mu - c\mathbb{1}_{\{F\le\theta\}}\right) F\left(V_F - f_y - \frac{\gamma_F}{2}g^2\right) + \frac{1}{2}\sigma^2 F^2\Big(V_{FF} - f_{yy} - 2f_{Fy} - \frac{\gamma_{FF}}{2}g^2 - \gamma(F)g_F^2 - 2\gamma_F g g_F\Big) - \frac{\lambda_{x+t}\gamma(F)}{2}\left(g - \max(F,G)\right)^2 - \lambda_{x+t}V + \lambda_{x+t}\max(F,G) = 0.$$

We plug in the derivatives of $V$, $f$, and $g$ in terms of $\nu^1$ and $\nu^2$ and verify that the equality still holds:

$$\begin{aligned}
&\nu_t^1 - \frac{\gamma(F)}{2}\left(\nu_t^2 - 2\nu^1\nu_t^1\right) + \left(\mu - c\mathbb{1}_{\{F\le\theta\}}\right)F\left(\nu_F^1 - \frac{\gamma(F)}{2}\nu_F^2 + \gamma(F)\nu^1\nu_F^1\right)\\
&+ \frac{1}{2}\sigma^2F^2\left(\nu_{FF}^1 - \frac{\gamma(F)}{2}\nu_{FF}^2 + \gamma(F)\nu^1\nu_{FF}^1\right) - \frac{\lambda_{x+t}\gamma(F)}{2}\left(\nu^1 - \max(F,G)\right)^2\\
&- \lambda_{x+t}\left[\nu^1 - \frac{\gamma(F)}{2}\left(\nu^2 - (\nu^1)^2\right)\right] + \lambda_{x+t}\max(F,G)\\
&= \left[\nu_t^1 + \left(\mu - c\mathbb{1}_{\{F<\theta\}}\right)F\nu_F^1 + \frac{1}{2}\sigma^2F^2\nu_{FF}^1 - \lambda_{x+t}\nu^1 + \lambda_{x+t}\max(F,G)\right]\\
&- \frac{\gamma(F)}{2}\left[\nu_t^2 + \left(\mu - c\mathbb{1}_{\{F<\theta\}}\right)F\nu_F^2 + \frac{1}{2}\sigma^2F^2\nu_{FF}^2 - \lambda_{x+t}\nu^2 + \lambda_{x+t}\max(F,G)^2\right]\\
&+ \gamma(F)\nu^1\left[\nu_t^1 + \left(\mu - c\mathbb{1}_{\{F<\theta\}}\right)F\nu_F^1 + \frac{1}{2}\sigma^2F^2\nu_{FF}^1 - \lambda_{x+t}\nu^1 + \lambda_{x+t}\max(F,G)\right]\\
&= 0.
\end{aligned}$$

We must now check the boundary conditions (18):

- for the terminal condition,

$$V(T,F,m) = \nu^1(T,F,m;\tau^{\hat{u}}) - \frac{\gamma(F)}{2}\left[\nu^2(T,F,m;\tau^{\hat{u}}) - (\nu^1(T,F,m;\tau^{\hat{u}})^2\right] = \max(F,G),$$

- for the condition when $F = 0$,

$$\begin{aligned}
&V_t - \frac{\lambda_{x+t}\gamma(0)}{2}\left(\nu^1 - G\right)^2 - \lambda_{x+t}V + \lambda_{x+t}G\\
&= \nu_t^1 - \frac{\gamma(0)}{2}\nu_t^2 + \gamma(0)\nu^1\nu_t^1 - \frac{\lambda_{x+t}\gamma(0)}{2}(\nu^1)^2 + \lambda_{x+t}\gamma(0)\nu^1G - \frac{\lambda_{x+t}\gamma(0)}{2}G^2\\
&- \lambda_{x+t}\left(\nu^1 - \frac{\gamma(0)}{2}(\nu^2 - (\nu^1)^2)\right) + \lambda_{x+t}G\\
&= \left(\nu^1 - \lambda_{x+t}\nu^1 + \lambda_{x+t}G\right) - \frac{\gamma(0)}{2}\left(\nu^2 - \lambda_{x+t}\nu^2 + \lambda_{x+t}G^2\right) + \gamma(0)\nu^1\left(\nu^1 - \lambda_{x+t}\nu^1 + \lambda_{x+t}G\right)\\
&= 0,
\end{aligned}$$

- and for the condition when F = m,

$$\begin{aligned}
&V_m - \alpha\mathbb{1}_{\{F\ge\theta\}}\left(V_F - f_y - \frac{\gamma'(F)}{2}(\nu^1)^2\right)\\
&= \nu_m^1 - \frac{\gamma(F)}{2}\left(\nu_m^2 - 2\nu^1\nu_m^1\right) - \alpha\mathbb{1}_{\{F\ge\theta\}}\left[\nu_F^1 - \frac{\gamma_F}{2}\left(\nu^2 - (\nu^1)^2\right) - \frac{\gamma(F)}{2}\left(\nu_F^2 - 2\nu^1\nu_F^1\right) + \frac{\gamma_F}{2}\nu^2 - \frac{\gamma_F}{2}(\nu^1)^2\right]\\
&= \nu_m^1 - \frac{\gamma(F)}{2}\left(\nu_m^2 - 2\nu^1\nu_m^1\right) - \alpha\mathbb{1}_{\{F\ge\theta\}}\left[\nu_F^1 - \frac{\gamma(F)}{2}\left(\nu_F^2 - 2\nu^1\nu_F^1\right)\right]\\
&= \left(\nu_m^1 - \alpha\mathbb{1}_{\{F\ge\theta\}}\nu_F^1\right) - \frac{\gamma(F)}{2}\left(\nu_m^2 - \alpha\mathbb{1}_{\{F\ge\theta\}}\nu_F^2\right) + \gamma(F)\nu^1\left(\nu_m^1 - \alpha\mathbb{1}_{\{F\ge\theta\}}\nu_F^1\right)\\
&= 0.
\end{aligned}$$

Finally, we check if the function *f* defined in the linear combination (27) solves Eq. (20) with boundary conditions (21). For ˆ*u* = 1,

$$\begin{aligned} f(t,F,m,y) &= \nu^1(t,F,m;\tau^{\hat{u}}) - \frac{\gamma(y)}{2}\nu^2(t,F,m;\tau^{\hat{u}}) \\ &= \max(F,G) - \frac{\gamma(y)}{2}\max(F,G)^2 \\ &= \psi(t,F) - \frac{\gamma(y)}{2}\psi(t,F). \end{aligned}$$

For ˆ*u* = 0, one obtains

$$f_t + \left(\mu - c\mathbb{1}_{\{F\leq\theta\}}\right) F f_F + \frac{1}{2}\sigma^2 F^2 f_{FF} - \lambda_{x+t} f + \lambda_{x+t}\left[\max(F,G) - \frac{\gamma(y)}{2}\max(F,G)^2\right] = 0.$$

As before, we plug in the derivatives of *f* in terms of $v^1$ and $v^2$ and verify that the equality still holds:

$$\begin{aligned} &\nu_t^1 - \frac{\gamma(y)}{2}\nu_t^2 + \left(\mu - c\mathbb{1}_{\{F\leq\theta\}}\right) F\left(\nu_F^1 - \frac{\gamma(y)}{2}\nu_F^2\right) + \frac{1}{2}\sigma^2 F^2\left(\nu_{FF}^1 - \frac{\gamma(y)}{2}\nu_{FF}^2\right) \\ &\quad - \lambda_{x+t}\left(\nu^1(t,F,m) - \frac{\gamma(y)}{2}\nu^2(t,F,m)\right) + \lambda_{x+t}\left[\max(F,G) - \frac{\gamma(y)}{2}\max(F,G)^2\right] \\ &\quad = \left[\nu_t^1 + \left(\mu - c\mathbb{1}_{\{F<\theta\}}\right) F\nu_F^1 + \frac{1}{2}\sigma^2 F^2 \nu_{FF}^1 - \lambda_{x+t}\nu^1 + \lambda_{x+t}\max(F,G)\right] \\ &\quad - \frac{\gamma(y)}{2}\left[\nu_t^2 + \left(\mu - c\mathbb{1}_{\{F<\theta\}}\right) F\nu_F^2 + \frac{1}{2}\sigma^2 F^2 \nu_{FF}^2 - \lambda_{x+t}\nu^2 + \lambda_{x+t}\max(F,G)^2\right] \\ &\quad = 0. \end{aligned}$$

Now we check the boundary conditions (21):

- for the terminal condition,

$$f(T,F,m,y) = \nu^1(T,F,m;\tau^{\hat{u}}) - \frac{\gamma(y)}{2}\nu^2(T,F,m;\tau^{\hat{u}}) = \max(F,G) - \frac{\gamma(y)}{2}\max(F,G)^2,$$

- for the condition when *F* = 0,

$$\begin{aligned} &f_t - \lambda_{x+t} f + \lambda_{x+t} G - \frac{\gamma(y)}{2}\lambda_{x+t} G^2 \\ &\quad = \nu_t^1 - \frac{\gamma(y)}{2}\nu_t^2 - \lambda_{x+t}\left(\nu^1 - \frac{\gamma(y)}{2}\nu^2\right) + \lambda_{x+t} G - \frac{\gamma(y)}{2}\lambda_{x+t} G^2 \\ &\quad = \left(\nu^1 - \lambda_{x+t}\nu^1 + \lambda_{x+t} G\right) - \frac{\gamma(y)}{2}\left(\nu^2 - \lambda_{x+t}\nu^2 + \lambda_{x+t} G\right) \\ &\quad = 0, \end{aligned}$$

- for the condition when $F = m$,

$$
\begin{aligned}
f_m - \alpha \mathbb{1}_{\{F \geq \theta\}} f_F &= \nu_m^1 - \frac{\gamma(y)}{2} \nu_m^2 - \alpha \mathbb{1}_{\{F \geq \theta\}} \left( \nu_F^1 - \frac{\gamma(y)}{2} \nu_F^2 \right) \\
&= \left( \nu_m^1 - \alpha \mathbb{1}_{\{F \geq \theta\}} \nu_F^1 \right) - \frac{\gamma(y)}{2} \left( \nu_m^2 - \alpha \mathbb{1}_{\{F \geq \theta\}} \nu_F^2 \right) \\
&= 0.
\end{aligned}
$$

This completes the proof. □